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by

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BOUNDARY FUNCTIONS

By Theodore John Kaczynski

Let H denote the set of all points in the Euclidean plane having positive y-coordinate, and let X denote the x-axis. If p is a point of X, then by an <u>arc at</u> p we mean a simple arc γ, having one endpoint at p, such that $\gamma - \{p\} \subseteq H$. Let f be a function mapping H into the Riemann sphere. By a <u>boundary function</u> for f we mean a function φ defined on a set $E \subseteq X$ such that for each $p \in E$ there exists an arc γ at p for which

$$\lim_{\substack{z \to p \\ z \in \gamma}} f(z) = \varphi(p).$$

The <u>set of curvilinear convergence of</u> f is the largest set on which a boundary function for f can be defined; in other words, it is the set of all points $p \in X$ such that there exists an arc at p along which f approaches a limit. A theorem of J. E. McMillan states that if f is a continuous function mapping H into the Riemann sphere, then the set of curvilinear convergence of f is of type $F_{\sigma\delta}$. In the first of the two chapters of this dissertation we give a more direct proof of this result than McMillan's, and we prove, conversely, that if A is a set of type $F_{\sigma\delta}$ in X, then there exists a bounded continuous complex-valued function in H having A as its set of curvi-

linear convergence. Next, we prove that a boundary function for a continuous function can always be made into a function of Baire class 1 by changing its values on a countable set of points. Conversely, we show that if φ is a function mapping a set $E \subseteq X$ into the Riemann sphere, and if φ can be made into a function of Baire class 1 by changing its values on a countable set, then there exists a continuous function in H having φ as a boundary function. (This is a slight generalisation of a theorem of Bagemihl and Piranian.) In the second chapter we prove that a boundary function for a function of Baire class $\xi \geq 1$ in H is of Baire class at most $\xi+1$. It follows from this that a boundary function for a Borel-measurable function is always Borel-measurable, but we show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable. The dissertation concludes with a list of problems remaining to be solved.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

INTRODUCTION

1. Preliminary Remarks

Let H denote the upper half-plane, and let X denote its frontier, the x-axis. If $x \in X$, then by an <u>arc at</u> x we mean a simple arc γ with one endpoint at x such that $\gamma - \{x\} \subseteq H$. Suppose that f is a function mapping H into some metric space Y. If E is any subset of X, we will say that a function $\varphi: E \to Y$ is a <u>boundary function</u> for f if, and only if, for each $x \in E$ there exists an arc γ at x such that

$$\lim_{\substack{z \to x \\ z \in \gamma}} f(z) = \varphi(x) .$$

The study of boundary functions in this degree of generality was initiated by Bagemihl and Piranian [2]. A function defined in H may have more than one boundary function defined on a given set $E \subseteq X$, but it follows from a famous theorem of Bagemihl [1] that any two such boundary functions differ on at most a countable set of points.

If f is defined in H, then the <u>set of curvilinear convergence</u> of f is the set of all points $x \in X$ such that there exists some arc at x along which f approaches a limit. Evidently, this is the largest set on which a boundary function for f can be defined. J. E. McMillan [10] discovered that the set of curvilinear convergence of a continuous function is always of type $F_{\sigma\delta}$, and in this paper we show that every set of type $F_{\sigma\delta}$ in X is the set of curvilinear

convergence of some continuous function. Next, we show that if φ is a function defined on a subset E of X, then φ is a boundary function for some continuous function if and only if φ can be made into a function of the first Baire class by changing its values on at most a countable set of points. (This solves a problem of Bagemihl and Piranian [2, Problem 1].) We then consider functions that are not assumed to be continuous, and we prove that a boundary function for a function of Baire class $\xi \geq 1$ is of Baire class at most $\xi + 1$ (thus proving another conjecture of Bagemihl and Piranian [2]). It follows from this that a boundary function for a Borel-measurable function is always Borel-measurable, and in the last section we show that a boundary function for a Lebesgue-measurable function need not be Lebesgue-measurable.

Most of the results appearing here have already been published ([8] and [9]). At the time I published these papers I did not expect to have to make use of this material for a dissertation.

2. Notation

R will denote the field of real numbers.

R^n will denote n-dimensional Euclidean space.

Points in R^n will be written in the form $\langle x_1, \ldots, x_n \rangle$ rather than $(x_1, \ldots, x_n)$ (to avoid confusion with open intervals of real numbers in the case n = 2).

If $v \in R^n$, then $|v|$ denotes the length of the vector v.

S^2 denotes $\{v \in R^3 : |v| = 1\}$. S^2 will be referred to as the <u>Riemann sphere</u>.

Let

$$H = \{ \langle x,y \rangle \in R^2 : y > 0 \}$$

$$H_n = \{ \langle x,y \rangle \in R^2 : \frac{1}{n} > y > 0 \}$$

$$X = \{ \langle x,0 \rangle : x \in R \}$$

$$X_n = \{ \left\langle x,\frac{1}{n} \right\rangle : x \in R \} .$$

We consider X as being identical with R. Thus, for example, $\langle x,0 \rangle \leq \langle x',0 \rangle$ means $x \leq x'$, and for p, q $\in$ X, the notations [p,q], [p,q), etc. refer to the obvious intervals on X.

If E is a subset of a topological space, then $\bar{E}$ denotes the closure of E, E^* denotes the interior of E, and E' denotes the complement of E. Of course, if E is a subset of X, then E^* means the interior of E relative to X, not relative to the whole plane. In Section 7, we often denote two line segments by s and s'. Since the prime notation is never used for complementation in that section, there is no danger of confusing s' with the complement of s.

If f is a function defined in a subset of R^2, then f(x,y) means $f(\langle x,y \rangle)$. Thus we write f(z) for $z \in R^2$ and f(x,y) for $x,y \in R$ interchangeably.

3. Baire Functions

In this section we review the main facts concerning Borel sets and Baire functions, and we prove some results that will be needed later.

If C is any family of sets, let C_δ be the family of all sets that can be written as a countable intersection of members of C, and

let C_σ be the family of all sets that can be written as a countable union of members of C.

Suppose M is a metrizable topological space. Let $P^1(M)$ be the family of all open subsets of M and let $Q^1(M)$ be the family of all closed subsets of M. If ξ is an ordinal number greater than 1, let

$$P^\xi(M) = (\bigcup_{\eta<\xi} Q^\eta(M))_\sigma$$

$$Q^\xi(M) = (\bigcup_{\eta<\xi} P^\eta(M))_\delta$$

For any ξ, $E \in Q^\xi(M) \Longleftrightarrow E' \in P^\xi(M)$.

For any subset L of M, $E \in P^\xi(L)$ (respectively $Q^\xi(L)$) if and only if there exists a set $D \in P^\xi(M)$ (respectively $Q^\xi(M)$) such that $E = D \cap L$.

$P^\xi(M)$ and $Q^\xi(M)$ are closed under finite unions and finite intersections. $P^\xi(M)$ is closed under countable unions and $Q^\xi(M)$ is closed under countable intersections.

If $\eta<\xi$, then $P^\eta(M) \cup Q^\eta(M) \subseteq P^\xi(M) \cap Q^\xi(M)$.

Let $F_\sigma(M)$ be the class of all F_σ sets of M, and let $G_\delta(M)$ be the class of all G_δ sets of M.

$$P^2(M) = F_\sigma(M) \text{ and } Q^2(M) = G_\delta(M).$$

Let Y be a metric space. For any family C of subsets of M we will say that a function $f : M \to Y$ is of <u>class</u> (C) if and only if $f^{-1}(U) \in C$ for every open set $U \subseteq Y$.

The following definition of the Baire classes is somewhat different from the classical definition, but it seems more convenient

for our purposes. A function $f : M \to Y$ is said to be of Baire class $0(M, Y)$ if and only if it is continuous. If ξ is an ordinal number greater than or equal to 1, then f is said to be of Baire class $\xi(M, Y)$ if and only if there exists a sequence of functions $\{f_n\}_{n=1}^{\infty}$ mapping M into Y, f_n being of Baire class $\eta_n(M,Y)$ for some $\eta_n < \xi$, such that $f_n \to f$ pointwise.

If $f : M \to Y$ is of Baire class $\xi(M, Y)$ and if L is a subset of M, then $f|_L$ is of Baire class $\xi(L, Y)$.

If K is a metric space, if $g : K \to M$ is continuous, and if $f : M \to Y$ is of Baire class $\xi(M, Y)$, then the composite function $f \circ g$ is of Baire class $\xi(K, Y)$.

If Y is separable and if $f : M \to Y$ is of Baire class $\xi(M, Y)$, then f is of class $(P^{\xi+1}(M))$ [4, page 294].

If Y is separable and arcwise connected, if $\xi \neq 1$, and if $f : M \to Y$ is of class $(P^{\xi+1}(M))$, then f is of Baire class $\xi(M, Y)$ [4].

For any ξ, if $f : M \to R$ is of class $(P^{\xi+1}(M))$, then f is of Baire class $\xi(M, R)$ [6].

If $L \in Q^{\xi+1}(M)$ and $f:L \to R$ is of Baire class $\xi(L, R)$, then f can be extended to a function $\overline{f} : M \to R$ of Baire class $\xi(M, R)$ [6].

We say that a function $f : M \to R$ is Borel measurable if, and only if, for every open set $U \subseteq R$, $f^{-1}(U)$ is a member of the σ-ring of subsets of M generated by the open sets.

If $f : M \to R$ is of some Baire class $\xi(M, R)$, then f is Borel-measurable, and, conversely, if $f : M \to R$ is Borel-measurable, then f is of Baire class $\xi(M, R)$ for some countable ordinal number ξ [7, page 294].

The proofs of Lemmas 1 through 6 are based on standard techniques in the study of Baire functions.

Lemma 1. Let M be a metric space, and let E and F be two F_σ sets in M. Then there exist two disjoint F_σ sets A and B $\subseteq$ M such that

$$E - F \subseteq A \qquad \text{and} \qquad F - E \subseteq B.$$

Proof. Let $E = \bigcup_{n=1}^{\infty} E_n$ and $F = \bigcup_{n=1}^{\infty} F_n$, where E_n and F_n are closed. Then

$$E_n, F_n \in F_\sigma(M) \cap G_\delta(M).$$

It is easy to check that $F_\sigma(M) \cap G_\delta(M)$ is an algebra (i.e., is closed under complementation, finite unions, and finite intersections). We inductively define a sequence of pairs of sets (A_n, B_n) as follows. Let

$$A_1 = E_1, \qquad B_1 = F_1 \cap A_1'.$$

For n > 1, let

$$A_n = E_n \cap \bigcap_{j=1}^{n-1} B_j', \qquad B_n = F_n \cap \bigcap_{j=1}^{n} A_j'.$$

By induction, $A_n, B_n \in F_\sigma(M) \cap G_\delta(M)$. Let

$$A = \bigcup_{n=1}^{\infty} A_n, \qquad B = \bigcup_{n=1}^{\infty} B_n.$$

Then A and B are F_σ sets. Notice that

$$\bigcup_{j=1}^{n-1} B_j \subseteq F \qquad \text{and} \qquad \bigcup_{j=1}^{n} A_j \subseteq E,$$

from which it follows that

$$A_n = E_n \cap \Big(\bigcup_{j=1}^{n-1} B_j\Big)' \supseteq E_n \cap F',$$

and

$$B_n = F_n \cap (\bigcup_{j=1}^{n} A_j)' \supseteq F_n \cap E' .$$

Therefore

$$A \supseteq \bigcup_{n=1}^{\infty} (E_n \cap F') = E - F$$

and

$$B \supseteq \bigcup_{n=1}^{\infty} (F_n \cap E') = F - E.$$

It only remains to show that $A \cap B = \phi$. Suppose $x \in A \cap B$. Choose ℓ, m with $x \in A_\ell$ and $x \in B_m$. If $\ell > m$, then $\ell > 1$, so that

$$A_\ell = E_\ell \cap \bigcap_{j=1}^{\ell-1} B_j' \subseteq B_m' .$$

Hence $x \in B_m'$ --a contradiction. On the other hand, if $\ell \leq m$, then

$$B_m = F_m \cap \bigcap_{j=1}^{m} A_j' \subseteq A_\ell' ,$$

so that $x \in A_\ell'$ --another contradiction. We conclude that $A \cap B = \phi$.■

If E is a subset of a space M, we let χ_E denote the characteristic function of E.

Lemma 2. Let L be a subset of a metric space M, and suppose that $E \in F_\sigma(L) \cap G_\delta(L)$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous real-valued functions on M such that $f_n \to \chi_E$ pointwise on L.

Proof. Both E and L - E are in $F_\sigma(L)$, so there exist sets E_1, $F_1 \in F_\sigma(M)$ such that

$$E = E_1 \cap L \quad \text{and} \quad L - E = F_1 \cap L.$$

By Lemma 1, there exist A, $B \in F_\sigma(M)$ such that $A \cap B = \phi$ and $E_1 - F_1 \subseteq A$, $F_1 - E_1 \subseteq B$. We have

$$E = A \cap L \qquad \text{and} \qquad L - E = B \cap L.$$

Write $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, where A_n, B_n are closed and $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$ for each n. By Urysohn's Lemma there exists a continuous function $f_n : M \to [0,1]$ such that

$$f_n(x) = 1 \qquad \text{when } x \in A_n$$

$$f_n(x) = 0 \qquad \text{when } x \in B_n .$$

$\{f_n\}_{n=1}^{\infty}$ is the desired sequence. ■

Lemma 3. Let L be a subset of a metric space M, $f : L \to R$ a function of class $(F_\sigma(L))$ that takes only finitely many different values. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous real-valued functions on M such that $f_n \to f$ pointwise on L.

Proof. From Banach's Hilfssatz 3 [4], we see that there exist real numbers $a_1, \ldots, a_q$ and sets

$$E_1, \ldots, E_q \in F_\sigma(L) \cap G_\delta(L)$$

such that

$$f = \sum_{j=1}^{q} a_j \chi_{E_j} .$$

If we choose for each j a sequence $\{f_n^j\}_{n=1}^{\infty}$ of continuous real-valued functions on M such that $f_n^j \xrightarrow[n]{} \chi_{E_j}$ pointwise on L, and if we set

$$f_n = \sum_{j=1}^{q} a_j f_n^j ,$$

then $\{f_n\}_{n=1}^{\infty}$ is the desired sequence. ■

Lemma 4. Let L be a metric space, f a bounded real-valued function on L of Baire class 1(L, R). Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$

of real-valued functions on L converging uniformly to f, such that each f_n is of class $(F_\sigma(L))$ and takes only finitely many different values.

Proof. f is of class $(F_\sigma(L))$ and the range of f is totally bounded, so an obvious modification of the proof of Banach's Hilfssatz 4 [4] gives the desired result.■

Lemma 5. Let M be a metric space, L a subset of M, $f : L \to R$ a function of Baire class $1(L, R)$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous real-valued functions on M such that $f_n \to f$ pointwise on L.

Proof. We first prove the lemma under the assumption that f is bounded. For any bounded real-valued function h, let

$$\|h\| = \sup \{ |h(x)| : x \in \text{domain of } h\} .$$

By Lemma 4 we can choose, for each n, a function $g_n : L \to R$ of class $(F_\sigma(L))$ such that g_n takes only finitely many different values and $\| g_n - f \| \leq \frac{1}{2^n}$. Let

$$h_1 = g_1 , \qquad h_n = g_n - g_{n-1} \text{ for } n > 1.$$

Then, for $n > 1$,

$$\|h_n\| = \|g_n - f + f - g_{n-1}\| \leq \frac{1}{2^n} + \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}} .$$

Each h_n is of class $(F_\sigma(L))$ and takes only finitely many different values, so by Lemma 3 we can choose (for each n) a sequence $\{h_n^j\}_{j=1}^{\infty}$ of continuous functions on M such that $h_n^j \underset{j}{\to} h_n$ pointwise on L.

Set

$$k_n^j(x) = -\|h_n\| \quad \text{if } h_n^j(x) \leq -\|h_n\|$$

$$k_n^j(x) = \|h_n\| \quad \text{if } h_n^j(x) \geq \|h_n\|$$

$$k_n^j(x) = h_n^j(x) \quad \text{if } -\|h_n\| < h_n^j(x) < \|h_n\| .$$

Then k_n^j is continuous, $k_n^j \underset{j}{\to} h_n$ pointwise on L, and $\|k_n^j\| \leq \|h_n\| < \frac{1}{2^{n-2}}$. Therefore, if we set

$$f_j = \sum_{n=1}^{\infty} k_n^j ,$$

then the series converges uniformly and f_j is continuous on M. We claim that $f_j \to f$ pointwise on L. Take any $x \in L$ and any $\varepsilon > 0$. Choose m large enough so that $\frac{1}{2^{m-2}} < \frac{1}{3}\varepsilon$. For each n, choose j(n) so that

$$j \geq j(n) \Longrightarrow |k_n^j(x) - h_n(x)| < \frac{1}{2^{n+1}} \frac{\varepsilon}{3} .$$

Let $i_o = \max\{j(1), \ldots, j(m)\}$. Then $j \geq i_o$ implies that

$$|f_j(x) - f(x)| \leq |f_j(x) - \sum_{n=1}^{m} k_n^j(x)| + |\sum_{n=1}^{m} k_n^j(x) - \sum_{n=1}^{m} h_n(x)|$$

$$+ |\sum_{n=1}^{m} h_n(x) - f(x)|$$

$$\leq \sum_{n=m+1}^{\infty} \|k_n^j\| + \sum_{n=1}^{m} |k_n^j(x) - h_n(x)| + \|g_m - f\|$$

$$\leq \frac{1}{2^{m-2}} + \left(\sum_{n=1}^{m} \frac{1}{2^{n+1}}\right) \frac{\varepsilon}{3} + \frac{1}{2^m} < 3\,\frac{\varepsilon}{3} = \varepsilon .$$

Thus $f_j(x) \underset{j}{\to} f(x)$ for each $x \in L$, and the lemma is proved for bounded f.

If f is not bounded, let

$$g(x) = \arctan f(x) \qquad (x \in L).$$

Then $-\frac{\pi}{2} < g(x) < \frac{\pi}{2}$ for every $x \in L$, and g is of Baire class $1(L, R)$, so there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of continuous functions on M converging to g pointwise on L. Set

$$h_n(x) = -\frac{\pi}{2} + \frac{1}{n} \quad \text{if} \quad g_n(x) \leq -\frac{\pi}{2} + \frac{1}{n}$$

$$h_n(x) = \frac{\pi}{2} - \frac{1}{n} \quad \text{if} \quad g_n(x) \geq \frac{\pi}{2} - \frac{1}{n}$$

$$h_n(x) = g_n(x) \quad \text{if} \quad -\frac{\pi}{2} + \frac{1}{n} < g_n(x) < \frac{\pi}{2} - \frac{1}{n}.$$

Then h_n is continuous on M, $-\frac{\pi}{2} < h_n(x) < \frac{\pi}{2}$, and $h_n \to g$ pointwise on L. Let $f_n(x) = \tan h_n(x)$. Then f_n is continuous on M and $f_n \to f$ pointwise on L. ∎

Lemma 6. If L is a subset of a metric space M and $f : L \to R^m$ is a function, then the following are equivalent.

(i) f is of Baire class $1(L, R^m)$.

(ii) f is of class $(F_\sigma(L))$.

(iii) There exists a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions mapping M into R^m such that $f_n \to f$ pointwise on L.

This lemma is an easy consequence of Lemma 5.

Definition. Let q be any point of R^3 lying inside the bounded open domain determined by S^2. By the <u>q-projection</u> of $R^3 - \{q\}$ onto S^2 we mean the function P_q defined as follows. If a is any point of $R^3 - \{q\}$, let ℓ be the unique ray, having its endpoint at q, that passes through a, and let $P_q(a)$ be the intersection point of ℓ with S^2. P_q is a continuous mapping of $R^3 - \{q\}$ onto S^2 that fixes every point of S^2.

Theorem 1. Let L be an arbitrary subset of R^2. Then a function $f : L \to S^2$ is of Baire class $1(L, S^2)$ if and only if it is of class $(F_\sigma(L))$.

Proof. Assume that $f : L \to S^2$ is of class $(F_\sigma(L))$. $S^2 \subseteq R^3$, so by Lemma 6 there exists a sequence $\{f_n\}_{n=1}^\infty$ of continuous functions mapping R^2 into R^3 such that $f_n \to f$ pointwise on L. Let

$$A_n = f_n^{-1}(\{v \in R^3 : |v| = \tfrac{1}{2}\})$$

$$B_n = f_n^{-1}(\{v \in R^3 : |v| \leq \tfrac{1}{2}\})$$

$$C_n = f_n^{-1}(\{v \in R^3 : |v| \geq \tfrac{1}{2}\}).$$

Let $f_n^{\,o} = f_n|_{A_n}$. According to [5, Lemma 2.9, page 299], $f_n^{\,o}$ can be extended to a continuous function $g_n : R^2 \to \{v \in R^3 : |v| = \frac{1}{2}\}$. Define $h_n : R^2 \to R^3 - \{0\}$ by setting

$$h_n(x) = g_n(x) \qquad \text{if} \quad x \in B_n$$

$$h_n(x) = f_n(x) \qquad \text{if} \quad x \in C_n.$$

Since B_n, C_n are closed, h_n is continuous, and it is easy to verify that $h_n(x) \to f(x)$ for each $x \in L$. Let $k_n : R^2 \to S^2$ be the composite function $P_o \circ h_n$. Then k_n is continuous, and for each $x \in L$, $k_n(x) \to P_o(f(x)) = f(x)$. Thus f is of Baire class $1(L, S^2)$. ■

Definition. Let M and Y be metric spaces. Then a function $f : M \to Y$ is said to be of <u>honorary Baire class</u> $2(M, Y)$ if and only if there exists a countable set $N \subseteq M$ and a function $g : M \to Y$ of Baire class $1(M, Y)$ such that $f(x) = g(x)$ for every $x \in M - N$.

Theorem 2. Let L be an arbitrary subset of R^2 and let Y be either the real line, a finite-dimensional Euclidean space, or S^2. Then a

function $f : L \to Y$ is of honorary Baire class $2(L, Y)$ if and only if there exists a countable set $N \subseteq L$ such that $f|_{L-N}$ is of class $(F_\sigma(L - N))$.

Proof. Suppose that $f : L \to Y$ is of honorary Baire class $2(L, Y)$. Then there exists $g : L \to Y$ of Baire class $1(L, Y)$ and a countable set $N \subseteq L$ such that $f|_{L-N} = g|_{L-N}$. But $g|_{L-N}$ is of class $(F_\sigma(L - N))$.

Conversely, suppose that $f|_{L-N}$ is of class $(F_\sigma(L - N))$, where N is countable. We must show that f is of honorary Baire class $2(L, Y)$. First consider the case where $Y = R^m$. Write

$$f(x) = \langle f_1(x), f_2(x), \ldots, f_m(x) \rangle .$$

Then $f_i|_{L-N}$ is of class $(F_\sigma(L - N))$ $(i=1,\ldots, m)$, and it follows that $f_i|_{L-N}$ is of Baire class $1(L - N, {}^*R)$. Since $L - N \in G_\delta(L)$, we can extend $f_i|_{L-N}$ to a function $g_i : L \to R$ of Baire class $1(L, R)$. If we set $g(x) = \langle g_1(x), \ldots, g_m(x) \rangle$, then g is of Baire class $1(L, R^m)$ and $g(x) = f(x)$ for $x \in L - N$, so we have the desired result.

Now consider the case where $Y = S^2$. Since $S^2 \subseteq R^3$, there exists, as we have just shown, a function $g : L \to R^3$ of Baire class $1(L, R^3)$ such that $g(x) = f(x)$ for all $x \in L - N$. Then $g(L) - S^2$ is countable, so there exists some point q in the bounded open domain determined by S^2 such that $q \notin g(L)$. Let h be the composite function $P_q \circ g$. Then h maps L into S^2, and for each $x \in L - N$,

$$h(x) = P_q(g(x)) = P_q(f(x)) = f(x).$$

If $U \subseteq S^2$ is open, then

$$h^{-1}(U) = g^{-1}(P_q^{-1}(U)) \in F_\sigma(L) ,$$

so h is of class $(F_\sigma(L))$. By Theorem 1, h is of Baire class $1(L, S^2)$, so we have the desired result.■

CHAPTER I

BOUNDARY FUNCTIONS FOR CONTINUOUS FUNCTIONS

If r is a positive number and if y_0 is a point of a metric space Y having metric ρ, then

$$S(r, y_0) \quad \text{denotes} \quad \{y \in Y : \rho(y, y_0) < r\}.$$

We will repeatedly make use of Theorem 11.8 on page 119 in [11] without making explicit reference to it. This theorem states that if D is a Jordan domain in R^2 or in $R^2 \cup \{\infty\}$, if γ is the frontier of D, and if α is a cross-cut in D whose endpoints divide γ into arcs γ_1 and γ_2, then $D-\alpha$ has two components, and the frontiers of these components are respectively $\alpha \cup \gamma_1$ and $\alpha \cup \gamma_2$. (The term cross-cut is defined on page 118 in [11].)

4. Domain of the Boundary Function

Definition. If f is a function mapping into a metric space Y, then the <u>set of curvilinear convergence</u> of f is defined to be

$$\{x \in X : \text{there exists an arc } \gamma \text{ at } x \text{ and there exists } y \in Y \text{ such that } \lim_{\substack{z \to x \\ z \in \gamma}} f(z) = y\}.$$

J. E. McMillan [10] proved that for suitable spaces Y, the set of curvilinear convergence of a continuous function is always of type $F_{\sigma\delta}$. We give a more direct proof of this result than McMillan's. (This proof can be modified to give a more general result; see [9].)

An interval of X will be called <u>nondegenerate</u> if and only if it contains more than one point.

Suppose γ is a cross-cut of H. If V is the bounded component of H - γ, let $L(\gamma) = \overline{V} \cap X$. Then $L(\gamma) = [c, d]$, where c and d are the endpoints of γ and $c < d$. Suppose Ω is a domain contained in H. Let Γ denote the family of all cross-cuts γ of H for which $\gamma \cap H \subseteq \Omega$, and let

$$I(\Omega) = \bigcup_{\gamma \in \Gamma} L(\gamma)^* .$$

Let acc(Ω) denote the set of all points on X that are accessible by arcs in Ω.

Lemma 7. Assume that acc (Ω) is nonempty. Let a be the infimum of acc (Ω) and let b be the supremum of acc (Ω). Then

$$I(\Omega) = (a, b) .$$

Proof. Suppose $x \in I(\Omega)$. Let γ be a cross-cut of H such that $\gamma \cap H \subseteq \Omega$ and $x \in L(\gamma)^*$. $L(\gamma) = [c, d]$, where c and d are the endpoints of γ and $c < d$. It is evident that c and d are in acc(Ω), so $a \leq c < x < d \leq b$, and $x \in (a, b)$. Conversely, suppose $x' \in (a, b)$. Then there exist points $c', d' \in \text{acc}(\Omega)$ with $c' < x' < d'$. Since Ω is arcwise connected, it is easy to show that there exists a cross-cut γ' of H, with $\gamma' \cap H \subseteq \Omega$, having c', d' as its endpoints. But then $x' \in (c', d') = L(\gamma')^*$, so $x' \in I(\Omega)$. ■

Lemma 8. If Ω_1 and Ω_2 are domains contained in H, and if

(1) $$I(\Omega_1) \cap \overline{\text{acc}(\Omega_1)} \quad \text{and} \quad I(\Omega_2) \cap \overline{\text{acc}(\Omega_2)}$$

are not disjoint, then Ω_1 and Ω_2 are not disjoint.

Proof. We assume that Ω_1 and Ω_2 are disjoint and derive a contradiction. Let a be a point in both of the two sets (1). Let γ_i be a cross-cut of H, with $\gamma_i \cap H \subseteq \Omega_i$, such that $a \in L(\gamma_i)^*$ $(i = 1, 2)$. Let U_i and V_i be the components of $H - \gamma_i$, where V_i is the bounded component. Observe that $\gamma_1 \cap H$ and $\gamma_2 \cap H$ are disjoint.

Suppose $\gamma_1 \cap H \subseteq V_2$ and $\gamma_2 \cap H \subseteq V_1$. Then, since $\gamma_1 \cap H \subseteq \overline{U}_1$, U_1 has a point in common with V_2. But, since U_1 is unbounded, U_1 cannot be contained in V_2, so U_1 must have a point in common with $\gamma_2 \cap H$. This contradicts the assumption that $\gamma_2 \cap H \subseteq V_1$, so we conclude that either $\gamma_1 \cap H \nsubseteq V_2$ or $\gamma_2 \cap H \nsubseteq V_1$. Hence, either $\gamma_1 \cap H \subseteq U_2$ or $\gamma_2 \cap H \subseteq U_1$. By symmetry, we may assume that $\gamma_2 \cap H \subseteq U_1$.

Ω_2 does not meet γ_1, and Ω_2 does meet U_1 (because $\gamma_2 \cap H \subseteq U_1 \cap \Omega_2$), so $\Omega_2 \subseteq U_1$. Since $a \in \overline{acc(\Omega_2)}$, there exists a point $b \in L(\gamma_1)^*$ such that $b \in acc(\Omega_2)$. But then $b \in \overline{\Omega}_2 \subseteq \overline{U}_1$, and this is impossible because the frontier of U_1 is disjoint from $L(\gamma_1)^*$. ■

Theorem 3 (J. E. McMillan). Let Y be a complete separable metric space and let $f : H \to Y$ be a continuous function. Then the set of curvilinear convergence of f is of type $F_{\sigma\delta}$.

Proof. Let $\{p_k\}_{k=1}^{\infty}$ be a countable dense subset of Y. Let $\{Q(n, m)\}_{m=1}^{\infty}$ be a counting of all sets of the form

$$\{\langle x, y\rangle : 0 < y < \frac{1}{n} \text{ and } r < t < r + \frac{1}{n}\}$$

where r is a rational number. Let $\{U(n, m, k, \ell)\}_{\ell=1}^{\infty}$ be a counting (with repetitions allowed) of the components of

$$f^{-1}(S(\frac{1}{2^n}, p_k)) \cap Q(n, m).$$

(We consider ϕ to be a component of ϕ.) Let

$$A(n, m, k, \ell) = \text{acc}[U(n, m, k, \ell)].$$

Set

$$B = \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} I(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}.$$

Since $I(U(n, m, k, \ell))$ is open in X it is of type F_σ. It follows that B is of type $F_{\sigma\delta}$. Let C denote the set of curvilinear convergence of f. I claim that $B \subseteq C$. Take any $b \in B$. For each n, choose $m[n]$, $k[n]$, $\ell[n]$ with

$$(2) \qquad b \in I(U(n, m[n], k[n], \ell[n])) \cap \overline{A(n, m[n], k[n], \ell[n])}$$
$$(n = 1, 2, 3, \ldots).$$

For convenience, set $U_n = U(n, m[n], k[n], \ell[n])$. By (2) and Lemma 8, U_n and U_{n+1} have some point z_n in common. For each n, we can choose an arc $\gamma_n \subseteq U_{n+1}$ with one endpoint at z_n and the other at z_{n+1}. Then $\gamma_n \subseteq Q(n+1, m[n+1])$. Also,

$$b \in \overline{A(n+1, m[n+1], k[n+1], \ell[n+1])} \subseteq \overline{U}_{n+1} \subseteq \overline{Q(n+1, m[n+1])},$$

and therefore each point of γ_n has distance less than $\frac{2}{n+1}$ from b. $\frac{2}{n+1} \to 0$ as $n \to \infty$; hence, if we set $\gamma = \{b\} \cup \bigcup_{n=1}^{\infty} \gamma_n$, then γ is an arc with one endpoint at b.

Since U_n and U_{n+1} have a point in common,

$$f^{-1}(S(\frac{1}{2^n}, p_{k[n]})) \text{ and } f^{-1}(S(\frac{1}{2^{n+1}}, p_{k[n+1]}))$$

have a common point, and hence

$$S(\frac{1}{2^n}, p_{k[n]}) \quad \text{and} \quad S(\frac{1}{2^{n+1}}, p_{k[n+1]})$$

have a common point. Therefore, if ρ is the metric on Y, then

$$\rho(p_{k[n]}, p_{k[n+1]}) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} < \frac{1}{2^{n-1}},$$

and therefore

$$\rho(p_{k[n]}, p_{k[n+r]}) \leq \sum_{i=1}^{r} \rho(p_{k[n+i-1]}, p_{k[n+i]}) < \sum_{i=1}^{r} \frac{1}{2^{n+i-2}} < \frac{1}{2^{n-2}}.$$

Thus $\{p_{k[n]}\}$ is a Cauchy sequence and must converge to some point $p \in Y$. Since

$$\gamma_n \subseteq U_{n+1} \subseteq f^{-1}(S(\frac{1}{2^{n+1}}, p_{k[n+1]})) \quad \text{and}$$

$$p_{k[n]} \underset{n}{\to} p,$$

$\lim\limits_{\substack{z \to b \\ z \in \gamma}} f(z) = p$. It is possible that γ is not a simple arc, but according to [12] we can replace γ by a simple arc $\gamma' \subseteq \gamma$. Thus $b \in C$, and we have shown that $B \subseteq C$.

Suppose $c \in C$. Let γ_o be an arc at c such that f approaches a limit p' along γ_o. Take any n. Choose k with $p' \in S(\frac{1}{2^n}, p_k)$. Choose m so that c is in the interior of $\overline{Q(n, m)} \cap X$. Then γ_o has a subarc γ_o', with one endpoint at c, such that

$$\gamma_o' - \{c\} \subseteq Q(n, m) \cap f^{-1}(S(\frac{1}{2^n}, p_k)).$$

Hence, for some ℓ, $c \in \text{acc}[U(n, m, k, \ell)] = A(n, m, k, \ell)$. This shows that

$$C \subseteq \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} A(n, m, k, \ell).$$

It is easy to deduce from Lemma 7 that the set

$$A(n, m, k, \ell) - I(U(n, m, k, \ell)) = A(n, m, k, \ell) - [I(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}]$$

contains at most two points. It follows by a routine argument that

$$\bigcap_n \bigcup_{m,k,\ell} A(n, m, k, \ell) - \bigcap_n \bigcup_{m,k,\ell} [I(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}]$$

is countable. Since

$$\bigcap_n \bigcup_{m,k,\ell} [I(U(n, m, k, \ell)) \cap \overline{A(n, m, k, \ell)}] = B \subseteq C$$

$$\bigcap_n \bigcup_{m,k,\ell} A(n, m, k, \ell),$$

$C - B$ is countable, and therefore C is of type $F_{\sigma\delta}$. ■

Next we will show that the foregoing theorem is as strong as possible, in this sense: if A is any set of type $F_{\sigma\delta}$ contained in X, then there exists a bounded continuous complex-valued function f defined in H such that A is the set of curvilinear convergence of f. The proof is unfortunately quite long.

Definition. Let E_1 and E_2 be two sets on the real line. A point p on the real line will be called a <u>splitting point</u> for E_1 and E_2 if either

$$x_1 \leq p \text{ for all } x_1 \in E_1 \quad \text{and} \quad p \leq x_2 \text{ for all } x_2 \in E_2$$

or $$x_2 \leq p \text{ for all } x_2 \in E_2 \quad \text{and} \quad p \leq x_1 \text{ for all } x_1 \in E_1.$$

We will say that two sets E_1 and E_2 <u>split</u>, or that E_1 <u>splits with</u> E_2, if and only if there exists a splitting point for E_1 and E_2.

Lemma 9. Let E be an F_σ set in R. Then there is a sequence $\{E_n\}_{n=1}^{\infty}$ of sets such that

(i) E_n is bounded and closed

(ii) if $n \neq m$, then either E_n and E_m are disjoint or E_n and E_m split

(iii) $E = \bigcup_{n=1}^{\infty} E_n$.

Proof. We can write $E = \bigcup_{n=1}^{\infty} A_n$ where A_n is closed, $A_n \subseteq A_{n+1}$ for all n, and $A_1 = \phi$.

Observe that if I is any open interval, then there exists a countable family $\{J_n\}_{n=1}^{\infty}$ of bounded closed intervals such that $n \neq m \Rightarrow J_n$ and J_m split, and $I = \bigcup_{n=1}^{\infty} J_n$. Since any open set of real numbers is a countable disjoint union of open intervals, it follows that for any open U there exists a countable family $\{I_n\}_{n=1}^{\infty}$ of bounded closed intervals such that $n \neq m \Rightarrow I_n$ and I_m split, and $U = \bigcup_{n=1}^{\infty} I_n$.

For each n, let $\{I_j^n\}_{j=1}^{\infty}$ be a family of bounded closed intervals such that $j \neq k \Rightarrow I_j^n$ and I_k^n split, and $A_n' = \bigcup_{j=1}^{\infty} I_j^n$. Let

$$\mathcal{F} = \{A_1\} \cup \{I_j^n \cap A_{n+1} : n = 1, 2, \ldots; j = 1, 2, \ldots\}.$$

Then $\mathcal{F}$ is a countable family of bounded closed sets, and

$$E = \bigcup_{n=1}^{\infty} A_n = A_1 \cup \bigcup_{n=1}^{\infty} (A_{n+1} \cap A_n')$$

$$= A_1 \cup \bigcup_{n=1}^{\infty} (A_{n+1} \cap \bigcup_{j=1}^{\infty} I_j^n) = A_1 \cup \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} (A_{n+1} \cap I_j^n)$$

$$= \bigcup \mathcal{F}.$$

Let F_1 and F_2 be any two distinct members of $\mathcal{F}$. If either F_1 or F_2 is $A_1 = \phi$, then F_1 and F_2 are automatically disjoint. If neither F_1 nor F_2 is A_1, then we can write

$$F_1 = I^{n(1)}_{j(1)} \cap A_{n(1)+1} \text{ and } F_2 = I^{n(2)}_{j(2)} \cap A_{n(2)+1}.$$

If $n(1) < n(2)$, then $n(1) + 1 \leq n(2)$, so

$F_2 = I^{n(2)}_{j(2)} \cap A_{n(2)+1} \subseteq A'_{n(2)} \subseteq A'_{n(1)+1}$, and therefore F_1 and F_2 are disjoint. A similar argument shows that if $n(2) < n(1)$, then F_1 and F_2 are disjoint. Thus, if F_1 and F_2 are not disjoint, then $n(1) = n(2)$ and we have

$$F_1 = I^n_{j(1)} \cap A_{n+1} \quad \text{and} \quad F_2 = I^n_{j(2)} \cap A_{n+1},$$

where $n = n(1) = n(2)$. But then $j(1) \neq j(2)$, so $I^n_{j(1)}$ and $I^n_{j(2)}$ split, and therefore F_1 and F_2 split. So we have shown that any two distinct members of $\mathcal{F}$ either split or are disjoint.

If $\mathcal{F}$ has infinitely many distinct members, let $E_1, E_2, E_3 \ldots$ be a counting of $\mathcal{F}$. If $\mathcal{F}$ has only finitely many distinct members, let $E_1, \ldots, E_m$ be the members of $\mathcal{F}$ and let $E_k = \phi$ for $k > m$. In either case, $\{E_n\}_{n=1}^{\infty}$ is the desired sequence. ■

If F is a closed subset of the real line, then by a <u>complementary interval</u> of F we mean a component of F'. (If F = R, then ϕ is considered to be a complementary interval of F.)

Definition. By a <u>special family</u> we mean a family $\mathcal{F}$ of subsets of R such that

(3) $\mathcal{F}$ is nonempty

(4) each member of $\mathcal{F}$ is bounded and closed

(5) there exists a sequence $\{F_n\}_{n=1}^{\infty}$ of members of $\mathcal{F}$ such that every member of $\mathcal{F}$ is equal to some F_n, and the following condition is satisfied.

(5a) If $m > n$, then either F_m is contained in one of the complementary

intervals of F_n, or else F_m splits with F_n.

Lemma 10. If E is an F_σ set in R, then there exists a special family $\mathcal{F}$ such that $E = \bigcup \mathcal{F}$.

Proof. By Lemma 9 we can choose a sequence $\{E_n\}_{n=1}^\infty$ of bounded closed sets such that if $n \neq m$ then E_n and E_m either split or are disjoint, and $E = \bigcup_{n=1}^\infty E_n$.

Let $n_1 = 1$ and let $F_1 = E_1$. Now suppose that $n_1, n_2, \ldots, n_s$ are chosen and $F_1, F_2, \ldots, F_{n_s}$ are chosen so that

(i) $1 = n_1 < n_2 < \ldots < n_s$

(ii) F_i is closed and bounded $(i = 1, \ldots, n_s)$

(iii) if $n_s \geq r > t \geq 1$, then either F_r is contained in one of the complementary intervals of F_t, or else F_r splits with F_t

(iv) if $1 \leq i \leq n_s$, then there exists $j \in \{1, \ldots, s\}$ such that $F_i \subseteq E_j$

(v) $\bigcup_{i=1}^{n_s} F_i = \bigcup_{i=1}^{s} E_i$.

We construct $F_{n_s+1}, \ldots, F_{n_{s+1}}$ as follows. Let $\mathcal{J}$ be the family of complementary intervals of the bounded closed set

$$\bigcup_{i=1}^{n_s} F_i = \bigcup_{i=1}^{s} E_i.$$

We assert that E_{s+1} meets at most finitely many members of $\mathcal{J}$. If this assertion is false, then there exists an infinite sequence $\{I_n\}_{n=1}^\infty$ of members of $\mathcal{J}$ such that $n \neq m$ implies $I_n \cap I_m = \phi$, and there exists (for each m) a point $x_m \in I_m \cap E_{s+1}$. $\{x_m\}_{m=1}^\infty$ is a bounded sequence, and $n \neq m$ implies that $x_n \neq x_m$. From this it follows that $\{x_m\}_{m=1}^\infty$ has either a strictly increasing or a strictly decreasing convergent

subsequence. We will assume that $\{x_{m(k)}\}_{k=1}^{\infty}$ is a strictly increasing convergent subsequence; the reasoning is similar in the case of a strictly decreasing convergent subsequence. Say $I_{m(k)} = (a_k, b_k)$. Then $a_k < x_{m(k)} < b_k$, so since $x_{m(k)} < x_{m(k+1)} < b_{k+1}$ and $x_{m(k)} \notin I_{m(k+1)}$ we must have $x_{m(k)} \leq a_{k+1} < x_{m(k+1)}$. Therefore, if we let

$$\bar{x} = \lim_{k\to\infty} x_{m(k)},$$

then $\bar{x} = \lim_{k\to\infty} a_k$ also. Moreover, for $k \geq 2$, a_k is a <u>finite</u> real number, so that $a_k \in \bigcup_{i=1}^{s} E_i$. Therefore there exists $u \in \{1, \ldots, s\}$ such that $a_k \in E_u$ for infinitely many values of k. Consequently $\bar{x} \in E_u$. But since $x_{m(k)} \in E_{s+1}$, $\bar{x} \in E_{s+1}$ also. But then $\bar{x} \in E_u \cap E_{s+1}$, so that E_u and E_{s+1} must split and $\bar{x}$ must be a splitting point for E_u and E_{s+1}. Since infinitely many a_k lie in E_u, E_u contains points that are less than $\bar{x}$; and E_{s+1} also contains points less than $\bar{x}$; therefore E_u and E_{s+1} cannot split, and we have a contradiction. This proves the assertion. Let

$$\mathcal{S} = \{\phi\} \cup \{\bar{I} \cap E_{s+1} : I \in \mathcal{J} \text{ and } I \cap E_{s+1} \neq \phi\}.$$

Let n_{s+1} equal n_s plus the number of members of $\mathcal{S}$. Let $F_{n_s+1}, \ldots, F_{n_{s+1}}$ be all the members of $\mathcal{S}$. We must show that conditions (i) through (v) are still satisfied when s is replaced by $s+1$. Conditions (i), (ii), and (iv) are obvious. The verification of (iii) is divided into three parts. Suppose $n_{s+1} \geq r > t \geq 1$.

Case I. Assume that $n_s \geq r > t \geq 1$. In this case we already know that either F_r is contained in one of the complementary intervals of F_t or else F_r splits with F_t.

Case II. Assume that $n_{s+1} \geq r > n_s \geq t \geq 1$. There exists $v \in \{1,\ldots,s\}$ such that $F_t \subseteq E_v$. Either E_v and E_{s+1} are disjoint or they split.

Case IIa. Assume E_v and E_{s+1} are disjoint. Either $F_r = \phi$ (in which case F_r is certainly contained in a complementary interval of F_t) or else $F_r \neq \phi$ and $F_r = \overline{I} \cap E_{s+1}$ for some $I \in \mathcal{Q}$. Let J be the smallest closed interval containing F_r. Then $J \subseteq \overline{I}$ and $J^* \subseteq I \subseteq (\bigcup_{i=1}^{s} E_i)'$, so that J^* does not meet E_v. The endpoints of J lie in $F_r \subseteq E_{s+1}$, so neither endpoint of J lies in E_v. So J does not meet E_v and therefore J does not meet F_t; from which it follows that F_r is contained in a complementary interval of F_t.

Case IIb. Assume that E_v and E_{s+1} split. Since $F_t \subseteq E_v$ and $F_r \subseteq E_{s+1}$ it follows that F_t and F_r split.

Case III. Assume that $n_{s+1} \geq r > t > n_s$. If either F_r or F_t is ϕ, it is clear that F_r is contained in a complementary interval of F_t. Otherwise, there exist $I_1, I_2 \in \mathcal{Q}$ such that $I_1 \cap I_2 = \phi$ and

$$F_r = \overline{I}_1 \cap E_{s+1} \quad \text{and} \quad F_t = \overline{I}_2 \cap E_{s+1}.$$

Since $\overline{I}_1$ and $\overline{I}_2$ evidently split, F_r and F_t must split.

Thus condition (iii) is verified.

As for (v), it is clear that

$$E_{s+1} - \bigcup_{i=1}^{s} E_i \subseteq \bigcup_{j=n_s+1}^{n_{s+1}} F_j \subseteq E_{s+1},$$

so that

$$\bigcup_{i=1}^{s+1} E_i = (\bigcup_{i=1}^{s} E_i) \cup (E_{s+1} - \bigcup_{i=1}^{s} E_i)$$

$$\subseteq (\bigcup_{j=1}^{n_s} F_j) \cup (\bigcup_{j=n_s+1}^{n_{s+1}} F_j) \subseteq (\bigcup_{i=1}^{s} E_i) \cup E_{s+1} = \bigcup_{i=1}^{s+1} E_i.$$

Hence $$\bigcup_{i=1}^{s+1} E_i = (\bigcup_{j=1}^{n_s} F_j) \cup (\bigcup_{j=n_s+1}^{n_{s+1}} F_j) = \bigcup_{j=1}^{n_{s+1}} F_j.$$

Thus we have shown that we can construct sequences $\{n_j\}_{j=1}^{\infty}$, $\{F_k\}_{k=1}^{\infty}$ in such a way that conditions (i) through (v) are satisfied for every value of s. If we set $\mathcal{F} = \{F_k : k = 1, 2, \ldots\}$, it is easy to verify that $\mathcal{F}$ is a special family and that $E = \bigcup \mathcal{F}$. ■

Definition. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two families of sets, let

$$\mathcal{F}_1 \wedge \mathcal{F}_2 = \{F_1 \cap F_2 : F_1 \in \mathcal{F}_1 \text{ and } F_2 \in \mathcal{F}_2\}.$$

Lemma 11. If $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are two special families, then $\mathcal{F} \wedge \tilde{\mathcal{F}}$ is a special family.

Proof. Conditions (3) and (4) in the definition of a special family are clearly satisfied, so we just have to verify (5).

Arrange all pairs of positive integers in a sequence according to the scheme shown in Figure 1. Let (a(k), b(k)) be the kth term of the sequence[1] (k = 1, 2, ...). Observe that $k < \ell$ if and

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[1]The reader may find it amusing to derive the following formulas for (a(k), b(k)). For real t, let [[t]] denote the largest integer that is strictly less than t. Then

$$a(k) = \frac{1}{2}\left(\left[\left[\frac{\sqrt{8k+1}+1}{2}\right]\right]^2 + \left[\left[\frac{\sqrt{8k+1}+1}{2}\right]\right]\right) - k + 1$$

$$= \frac{1}{8}\left([[\sqrt{8k+1}]] + \frac{5}{2} - \frac{1}{2}(-1)^{[[\sqrt{8k+1}]]}\right)\left([[\sqrt{8k+1}]] + \frac{1}{2} - \frac{1}{2}(-1)^{[[\sqrt{8k+1}]]}\right) - k + 1$$

$$= \begin{cases} \frac{1}{8}([[\sqrt{8k+1}]] + 3)([[\sqrt{8k+1}]] + 1) - k + 1 & \text{if } [[\sqrt{8k+1}]] \text{ is odd} \\ \frac{1}{8}([[\sqrt{8k+1}]] + 2)[[\sqrt{8k+1}]] - k + 1 & \text{if } [[\sqrt{8k+1}]] \text{ is even,} \end{cases}$$

and

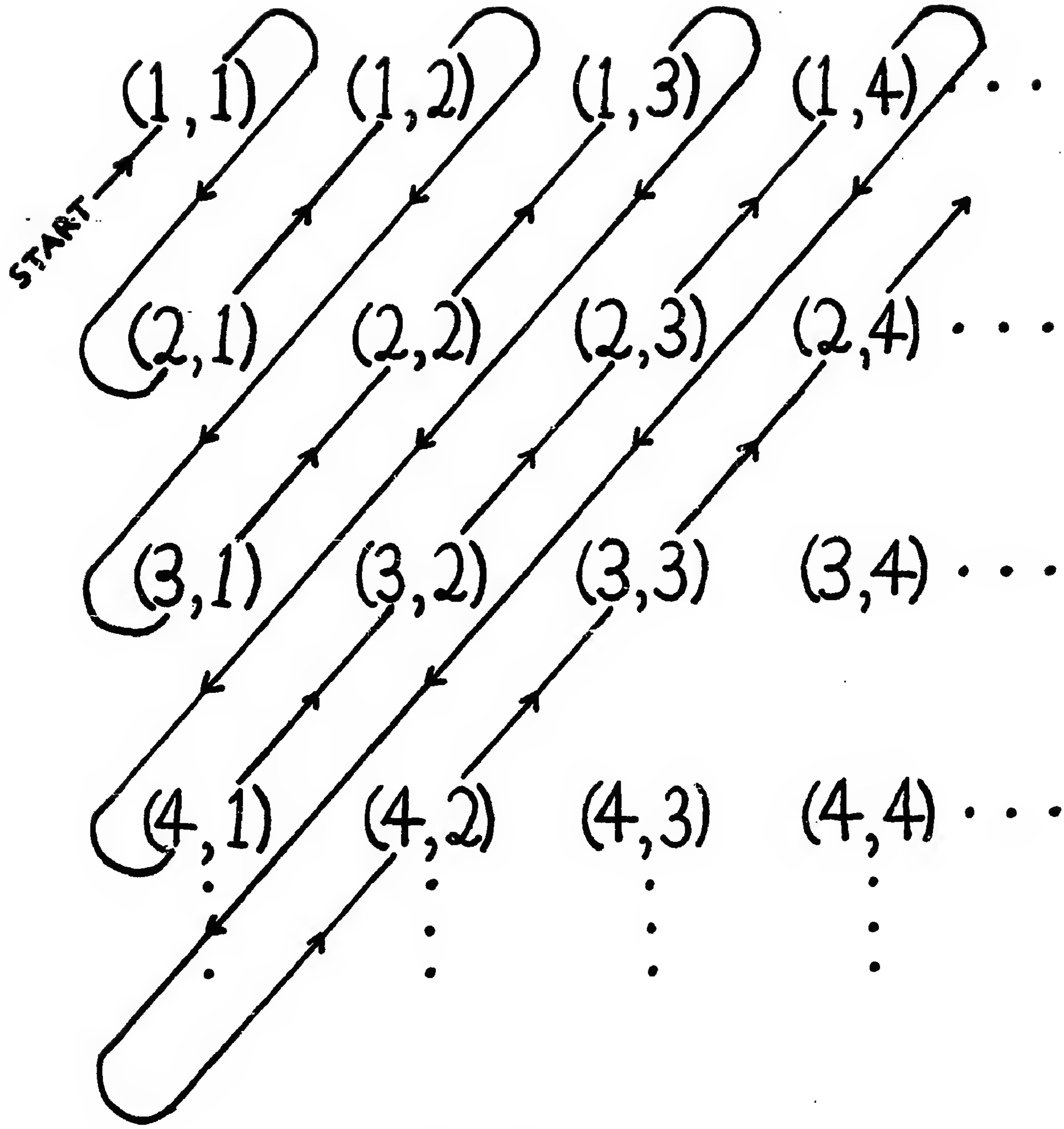

Figure 1.

only if either $a(k) + b(k) < a(\ell) + b(\ell)$ or else $a(k) + b(k) = a(\ell) + b(\ell)$ and $b(k) < b(\ell)$. Thus $k < \ell$ implies that either $a(k) < a(\ell)$ or $b(k) < b(\ell)$.

Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathcal{F}$ such that every member of $\mathcal{F}$ is equal to some F_n and such that condition (5a) in the definition of a special family is satisfied. Let $\{\tilde{F}_n\}_{n=1}^{\infty}$ be a similar sequence for $\tilde{\mathcal{F}}$. Set

$$\hat{F}_k = F_{a(k)} \cap \tilde{F}_{b(k)}.$$

Then $\{\hat{F}_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{F} \wedge \tilde{\mathcal{F}}$ such that every member of $\mathcal{F} \wedge \tilde{\mathcal{F}}$ is equal to some $\hat{F}_k$. We must show that condition (5a) is satisfied. Suppose that $\ell > k$. Two cases occur.

Case I. $a(k) < a(\ell)$.

Note that $\hat{F}_k \subseteq F_{a(k)}$ and $\hat{F}_\ell \subseteq F_{a(\ell)}$. Either $F_{a(\ell)}$ is contained in one of the complementary intervals of $F_{a(k)}$ (in which case $\hat{F}_\ell$ is contained in a complementary interval of $\hat{F}_k$), or else $F_{a(\ell)}$ and $F_{a(k)}$ split (in which case $\hat{F}_\ell$ and $\hat{F}_k$ split).

Case II. $b(k) < b(\ell)$.

In this case a similar argument shows that either $\hat{F}_\ell$ is contained in

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$$b(k) = \frac{1}{2}([[\frac{\sqrt{8k+1}+1}{2}]] - [[\frac{\sqrt{8k+1}+1}{2}]]^2) + k$$

$$= \frac{-1}{8}([[\sqrt{8k+1}]] + \frac{1}{2} - \frac{1}{2}(-1)^{[[\sqrt{8k+1}]]})([[\sqrt{8k+1}]] - \frac{3}{2} - \frac{1}{2}(-1)^{[[\sqrt{8k+1}]]}) + k$$

$$= \begin{cases} \frac{-1}{8}([[\sqrt{8k+1}]] + 1)([[\sqrt{8k+1}]] - 1) + k & \text{if } [[\sqrt{8k+1}]] \text{ is odd} \\ \frac{-1}{8}[[\sqrt{8k+1}]]([[\sqrt{8k+1}]] - 2) + k & \text{if } [[\sqrt{8k+1}]] \text{ is even.} \end{cases}$$

a complementary interval of $\hat{F}_k$ or $\hat{F}_\ell$ and $\hat{F}_k$ split. Thus condition (5a) is satisfied, and $\mathcal{F}\wedge\tilde{\mathcal{F}}$ is a special family. ■

Lemma 12. Let E_1, E_2 be two F_σ sets in R such that $E_1 \subseteq E_2$, and suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are special families such that $E_1 = \bigcup\mathcal{F}_1$ and $E_2 = \bigcup\mathcal{F}_2$. Then $E_1 = \bigcup(\mathcal{F}_1 \wedge \mathcal{F}_2)$.

The proof is obvious.

Next we introduce some notation.

Let J be a nonempty interval on X with endpoints a, b $(a \leq b)$. By Trap (J, ε, θ) (where $\theta \in (0, \frac{\pi}{2})$ and $\varepsilon > 0$) we mean the interior of the trapezoid shown in Figure 2. That is,

$$\mathrm{Trap}(J, \varepsilon, \theta) = \{ \langle x, y \rangle \ : 0 < y < \varepsilon,\ a + y \operatorname{ctn} \theta < x < b - y \operatorname{ctn} \theta\}.$$

For $\theta \in (0, \frac{\pi}{2})$, let Tri (J, θ) be the closed triangular area shown in Figure 3. That is,

$$\mathrm{Tri}\ (J, \theta) = \{ \langle x, y \rangle \ : y \geq 0 \text{ and } a + y \operatorname{ctn} \theta \leq x \leq b - y \operatorname{ctn} \theta\}.$$

If $x_0 \in X$, $\varepsilon > 0$, and $\theta \in (0, \frac{\pi}{2})$, let $S(x_0, \varepsilon, \theta)$ denote the open Stolz angle shown in Figure 4. That is,

$$S(x_0, \varepsilon, \theta) = \{ \langle x, y \rangle \ : 0 < y < \varepsilon,\ x_0 + y \operatorname{ctn} (\pi - \theta) < x < x_0 + y \operatorname{ctn} \theta\}.$$

If K is a closed set on a real line, let J(K) be the smallest closed interval containing K. If K is bounded, closed, and nonempty, $\varepsilon > 0$, and $0 < \beta < \alpha < \frac{\pi}{2}$, then we define

$$B(K, \varepsilon, \alpha, \beta) = \mathrm{Trap}\ (J(K), \varepsilon, \alpha) - \bigcup_{I \in \mathcal{J}} \mathrm{Tri}\ (I, \beta),$$

where $\mathcal{J}$ denotes the set of complementary intervals of K.

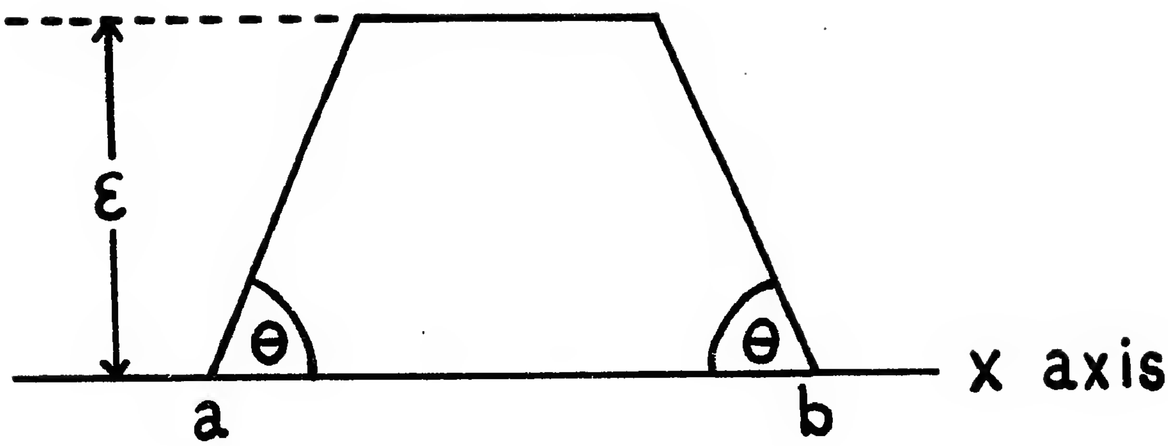

Figure 2.--Trap(J, ε , θ)

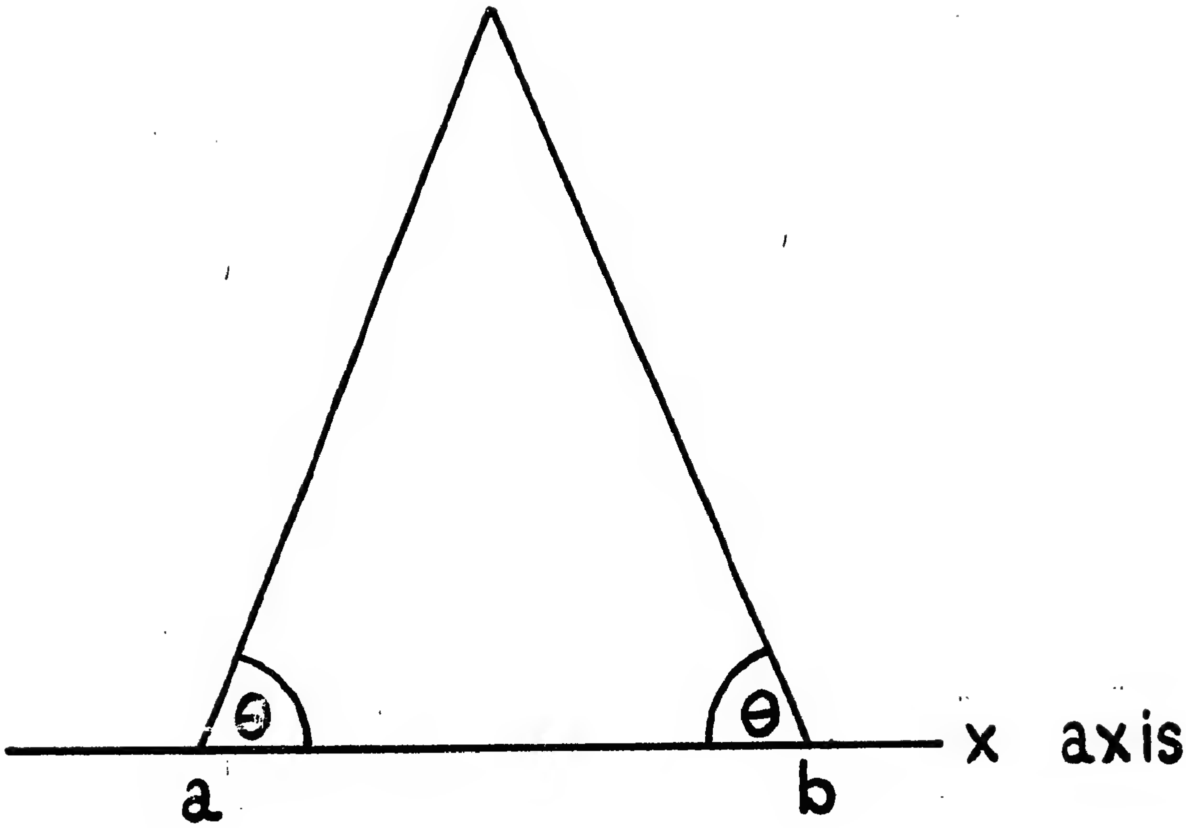

Figure 3.--Tri(J, Θ)

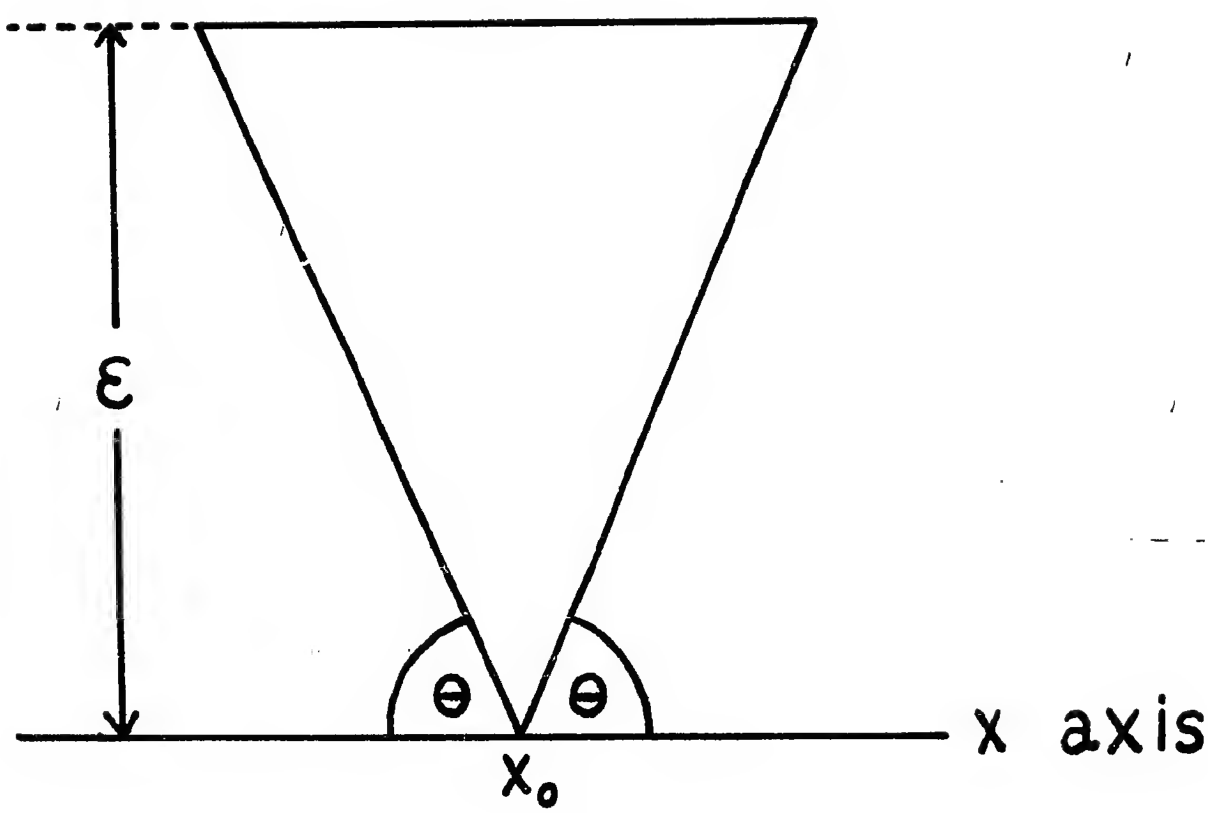

Figure 4.--$S(x_0, \varepsilon, \theta)$

We state without proof the following readily verifiable facts.

(6) $B(K, \varepsilon, \alpha, \beta)$ is an open subset of H.

(7) $S(e, \varepsilon, \theta)$ is an open subset of H.

(8) If K_1 and K_2 split, then for any ε_1, ε_2, α, β,

$$B(K_1, \varepsilon_1, \alpha, \beta) \quad \text{and} \quad B(K_2, \varepsilon_2, \alpha, \beta)$$

are disjoint.

(9) Suppose that $K_1 \subseteq K$, $\varepsilon > \varepsilon_1 > 0$, and $0 < \beta < \beta_1 < \alpha_1 < \alpha \leq \frac{\pi}{2}$. Then

$$\overline{B(K_1, \varepsilon_1, \alpha_1, \beta_1)} \cap H \subseteq B(K, \varepsilon, \alpha, \beta).$$

(10) Suppose K_1 is contained in one of the complementary intervals of K, and suppose ε, α, β are given. Then there exists $\delta > 0$ such that for every $\eta \leq \delta$,

$$B(K, \varepsilon, \alpha, \beta) \quad \text{and} \quad B(K_1, \eta, \alpha, \beta)$$

are disjoint.

(11) Suppose that $\alpha < \theta < \frac{\pi}{2}$ and $x_0 \notin J(K)^*$. Then, for any ε, ε_1, β,

$$B(K, \varepsilon, \alpha, \beta) \quad \text{and } S(x_0, \varepsilon_1, \theta)$$

are disjoint.

(12) Suppose that $x_0 \in K \cap J(K)^*$ and $\beta < \alpha < \theta < \frac{\pi}{2}$. Let ε be given. Then there exists $\delta > 0$ such that for every $\eta \leq \delta$,

$$\overline{S(x_0, \eta, \theta)} \cap H \subseteq B(K, \varepsilon, \alpha, \beta).$$

(13) Suppose that $\varepsilon < \varepsilon'$ and $\theta' < \theta$. Then

$$\overline{S(x_0, \varepsilon, \theta)} \cap H \subseteq S(x_0, \varepsilon', \theta').$$

(14) Suppose $x_o \notin K$ and ε, α, β, θ are given. Then there exists $\delta > 0$ such that for every $\eta \leq \delta$,

$$S(x_o, \eta, \theta) \quad \text{and} \quad B(K, \varepsilon, \alpha, \beta)$$

are disjoint.

(15) If $x_o \neq x_1$ and ε, θ are given, then there exists $\delta > 0$ such that for every $\eta \leq \delta$,

$$S(x_o, \varepsilon, \theta) \quad \text{and} \quad S(x_1, \eta, \theta)$$

are disjoint.

(16) $\overline{B(K, \varepsilon, \alpha, \beta)} \cap X \subseteq K$.

(17) $\overline{S(x_o, \varepsilon, \theta)} \cap X = \{x_o\}$.

Definition. If $\mathcal{F}$ is a special family, let $\mathcal{F}^2$ be the set of all members of $\mathcal{F}$ that have two or more points.

Definition. Let $\mathcal{F}$ be a special family, let E be the set of all end-points of intervals J(F) where $F \in \mathcal{F}$, $F \neq \phi$, and suppose that $0 < \beta < \alpha < \theta < \frac{\pi}{2}$. By a <u>pair of special</u> α, β, θ <u>functions</u> for $\mathcal{F}$ we mean a pair (ε, δ), where ε and δ are positive real-valued functions, the domain of ε is E, the domain of δ is $\mathcal{F}^2$, and

(18) for each $\eta > 0$, there exist at most finitely many $F \in \mathcal{F}^2$ such that $\delta(F) \geq \eta$;

(19) for each $\eta > 0$, there exist at most finitely many $e \in E$ such that $\varepsilon(e) \geq \eta$;

(20) if $e, e' \in E$ and $e \neq e'$, then

$$S(e, \varepsilon(e), \theta) \quad \text{and} \quad S(e', \varepsilon(e'), \theta)$$

are disjoint;

(21) if $F, K \in \mathcal{F}^2$ and $F \neq K$, then

$$B(F, \delta(F), \alpha, \beta) \quad \text{and} \quad B(K, \delta(K), \alpha, \beta)$$

are disjoint;

(22) if $e \in E$ and $F \in \mathcal{F}^2$, then

$$S(e, \varepsilon(e), \theta) \quad \text{and} \quad B(F, \delta(F), \alpha, \beta)$$

are disjoint.

Lemma 13. Let $\mathcal{F}$ be a special family and suppose that $0 < \beta < \alpha < \theta < \frac{\pi}{2}$. Then there exists a pair of special α, β, θ functions for $\mathcal{F}$.

Proof. Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of members of $\mathcal{F}$ of the type referred to in condition (5) in the definition of a special family. Let

$$\mathcal{F}^2(n) = \{F \in \mathcal{F}^2 : F = F_k \text{ for some } k \leq n\}$$

E = set of all endpoints of intervals $J(F)$ for $F \in \mathcal{F}$, $F \neq \phi$

$E(n) = \{e \in E : e$ is an endpoint of $J(F_k)$ for some $k \leq n$ for which $F_k \neq \phi\}$.

If $J(F_1)$ has one endpoint e, set $\varepsilon(e) = 1$. If $J(F_1)$ has two endpoints e_1, e_2, then by (15) we can choose $\varepsilon(e_1) \leq 1$ and $\varepsilon(e_2) \leq 1$ so that $S(e_1, \varepsilon(e_1), \theta)$ and $S(e_2, \varepsilon(e_2), \theta)$ are disjoint. If $F_1 \in \mathcal{F}^2$, set $\delta(F_1) = 1$. In this case, $J(F_1)$ has two endpoints e_1 and e_2 and (by (11)) $B(F_1, \delta(F_1), \alpha, \beta)$, $S(e_1, \varepsilon(e_1), \theta)$ and $S(e_2, \varepsilon(e_2), \theta)$ are all disjoint.

Now suppose that $\varepsilon(e)$ and $\delta(F)$ have been defined for all

$e \in E(n)$ and all $F \in \mathcal{F}^2(n)$ in such a way that

(i) if $e, e' \in E(n)$ and $e \neq e'$, then $S(e, \varepsilon(e), \theta)$ and $S(e', \varepsilon(e'), \theta)$ are disjoint;

(ii) if $F, K \in \mathcal{F}^2(n)$ and $F \neq K$, then $B(F, \delta(F), \alpha, \beta)$ and $B(K, \delta(K), \alpha, \beta)$ are disjoint;

(iii) if $e \in E(n)$ and $F \in \mathcal{F}^2(n)$, then $S(e, \varepsilon(e), \theta)$ and $B(F, \delta(F), \alpha, \beta)$ are disjoint;

(iv) if $e \in E(n)$ and $k \leq n$ is the least integer for which $e \in E(k)$, then $\varepsilon(e) \leq \frac{1}{k}$;

(v) if $F \in \mathcal{F}^2(n)$ and $k \leq n$ is the least integer for which $F \in \mathcal{F}^2(k)$, then $\delta(F) \leq \frac{1}{k}$.

We must extend the definitions of ε and δ to $E(n+1)$ and $\mathcal{F}^2(n+1)$ in such a way that conditions (i) through (v) are still satisfied when n is replaced by n+1.

Case I. If $F_{n+1} = \phi$ or if $F_{n+1} = F_k$ for some $k \leq n$, then $\mathcal{F}^2(n+1) = \mathcal{F}^2(n)$ and $E(n+1) = E(n)$, so that nothing is required to be done.

Case II. If F_{n+1} consists of a single point e and if $e \in F_k$ for some $k \leq n$, then (since F_{n+1} and F_k must split in this case) e is an endpoint of $J(F_k)$, so that again $\mathcal{F}^2(n+1) = \mathcal{F}^2(n)$ and $E(n+1) = E(n)$, and nothing is required to be done.

Case III. Suppose that F_{n+1} consists of a single point e_o and that for each $k \leq n$, $e_o \notin F_k$. By (14), (15), and the fact that $E(n)$ and $\mathcal{F}^2(n)$ are finite, we can choose $\varepsilon(e_o) \in (0, \frac{1}{n+1})$ so that $S(e_o, \varepsilon(e_o), \theta)$ is disjoint from $S(e, \varepsilon(e), \theta)$ and from $B(F, \delta(F), \alpha, \beta)$ for each $e \in E(n)$ and each $F \in \mathcal{F}(n)$. The construction is then finished for

$E(n+1)$ and $\mathcal{F}(n+1)$.

Case IV. Suppose that F_{n+1} contains at least two points and that, for each $k \leq n$, $F_k \nsubseteq F_{n+1}$. For each $k \leq n$, either F_{n+1} splits with F_k, or else F_{n+1} is contained in a complementary interval of F_k. Since $\mathcal{F}^2(n)$ is finite, (8) and (10) show that we can choose $\delta(F_{n+1}) \in (0, \frac{1}{n+1})$ so that $B(F_{n+1}, \delta(F_{n+1}), \alpha, \beta)$ is disjoint from $B(F, \delta(F), \alpha, \beta)$ for each $F \in \mathcal{F}^2(n)$.

Say $e \in E(n)$. Then e is an endpoint of $J(F_k)$ for some $k \leq n$, so (since F_{n+1} either splits with F_k or is contained in a complementary interval of F_k) $e \notin J(F_{n+1})^*$. By (11), $B(F_{n+1}, \delta(F_{n+1}), \alpha, \beta)$ and $S(e, \varepsilon(e), \theta)$ are disjoint.

Let e_o, e_o' be the endpoints of $J(F_{n+1})$.

Case IVa. $e_o, e_o' \in E(n)$.

In this case the construction is already finished.

Case IVb. $e_o \in E(n)$ and $e_o' \notin E(n)$.

If $e_o' \in F_k$ for some $k \leq n$, then F_{n+1} splits with F_k, so that e_o' must be and endpoint of $J(F_k)$ --which contradicts the assumption that $e_o' \notin E(n)$. Hence, for each $k \leq n$, $e_o' \notin F_k$. By (14), (15), and the fact that $E(n)$ and $\mathcal{F}^2(n)$ are finite, we can choose $\varepsilon(e_o') \in (0, \frac{1}{n+1})$ so that $S(e_o', \varepsilon(e_o'), \theta)$ is disjoint from $S(e, \varepsilon(e), \theta)$ and from $B(F, \delta(F), \alpha, \beta)$ for each $e \in E(n)$ and each $F \in \mathcal{F}^2(n)$. By (11), $S(e_o', \varepsilon(e_o'), \theta)$ and $B(F_{n+1}, \delta(F_{n+1}), \alpha, \beta)$ are disjoint. Thus the construction is finished for $E(n+1)$ and $\mathcal{F}^2(n+1)$.

Case IVc. $e_o \notin E(n)$ and $e_o' \in E(n)$.

This case is essentially the same as Case IVb.

Case IVd. $e_o \notin E(n)$ and $e_o' \notin E(n)$.

If $e_o \in F_k$ for some $k \leq n$, then F_{n+1} splits with F_k, so e_o is an endpoint of $J(F_k)$; a contradiction. Thus $e_o \notin F_k$ for $k \leq n$, and similarly $e_o' \notin F_k$ for $k \leq n$. Therefore, by (14) and (15), we can choose $\varepsilon(e_o)$ and $\varepsilon(e_o') \in (0, \frac{1}{n+1})$ so that $S(e_o, \varepsilon(e_o), \theta)$ and $S(e_o', \varepsilon(e_o'), \theta)$ are disjoint and each of $S(e_o, \varepsilon(e_o), \theta)$ and $S(e_o', \varepsilon(e_o'), \theta)$ is disjoint from every $S(e, \varepsilon(e), \theta)$ $(e \in E(n))$ and from every $B(F, \delta(F), \alpha, \beta)$ $(F \in \mathcal{F}^2(n))$. By (11), $S(e_o, \varepsilon(e_o), \theta)$ and $S(e_o', \varepsilon(e_o'), \theta)$ are each disjoint from $B(F_{n+1}, \delta(F_{n+1}), \alpha, \beta)$, so the construction is finished for $E(n+1)$ and $\mathcal{F}^2(n+1)$.

We have shown that we can inductively define $\varepsilon(e)$ for every $e \in E$ and $\delta(F)$ for every $F \in \mathcal{F}^2$ in such a way that (i) through (v) are satisfied for every value of n. Conditions (20), (21) and (22) in the definition of a pair of α, β, θ special functions are thus automatically satisfied by (ε, δ). We must verify that (18) and (19) are also satisfied.

Suppose (19) is false. Then there exists $\eta > 0$ and there exists an infinite sequence $\{e_k\}_{k=1}^{\infty}$ of distinct members of E such that $\varepsilon(e_k) \geq \eta$ for every k. Let m(k) be the least integer for which e_k is an endpoint of $J(F_{m(k)})$. Each $J(F_m)$ has at most two endpoints, so, since the e_k are all distinct, there exists (for given m) at most two values of k for which $m(k) = m$. Therefore there exist infinitely many distinct integers among m(1), m(2), m(3), ... Consequently there exists j with $\frac{1}{m(j)} < \eta$. But, by (iv), $\varepsilon(e_j) \leq \frac{1}{m(j)} < \eta$, a contradiction. So (19) must be true. A similar argument shows that (18) is true. ■

Lemma 14. Let $\mathcal{F}$ be a special family, $0 \le \beta < \alpha < \theta < \frac{\pi}{2}$, and let E be the set of all endpoints of intervals J(F) for $F \in \mathcal{F}$. Suppose (ε, δ) is a pair of special α, β, θ functions for $\mathcal{F}$. If ε_1, δ_1 are two real-valued functions having domains E and $\mathcal{F}^2$ respectively, and if

$$0 < \varepsilon_1(e) \le \varepsilon(e) \quad \text{for all } e \in E, \text{ and}$$
$$0 < \delta_1(F) \le \delta(F) \quad \text{for all } F \in \mathcal{F}^2,$$

then $(\varepsilon_1, \delta_1)$ is a pair of special α, β, θ functions for $\mathcal{F}$.

Proof. This follows from the fact that

$$S(x_0, \varepsilon', \theta) \subseteq S(x_0, \varepsilon'', \theta)$$

and $$B(K, \varepsilon', \alpha, \beta) \subseteq B(K, \varepsilon'', \alpha, \beta)$$

whenever $\varepsilon' \le \varepsilon''$. ∎

Theorem 4. Let A be any set of type $F_{\sigma\delta}$ in X. Then there exists a bounded continuous complex-valued function f defined in H such that A is the set of curvilinear convergence of f.

Proof. We can write $A = \bigcap_{n=1}^{\infty} A_n$, where each A_n is of type F_σ and $A_{n+1} \subseteq A_n$ for every n. For each n, let $\mathcal{F}_n$ be a special family with $\bigcup \mathcal{F}_n = A_n$. Let

$$\hat{\mathcal{F}}_1 = \mathcal{F}_1$$
$$\hat{\mathcal{F}}_{n+1} = \hat{\mathcal{F}}_n \wedge \mathcal{F}_{n+1} \quad \text{for } n > 1.$$

By Lemmas 11 and 12, together with mathematical induction, $\hat{\mathcal{F}}_n$ is a special family and $\bigcup \hat{\mathcal{F}}_n = A_n$. Moreover, every member of $\hat{\mathcal{F}}_{n+1}$ is a subset of some member of $\hat{\mathcal{F}}_n$.

Let $\{\beta_n\}_{n=1}^{\infty}$ be a strictly ascending sequence in $(0, \frac{\pi}{8})$ converging to $\frac{\pi}{8}$.

Let $\{\alpha_n\}_{n=1}^{\infty}$ be a strictly descending sequence in $(\frac{\pi}{8}, \frac{\pi}{4})$ converging to $\frac{\pi}{8}$.

Let $\{\theta_n\}_{n=1}^{\infty}$ be a strictly ascending sequence in $(\frac{\pi}{4}, \frac{3\pi}{8})$ converging to $\frac{3\pi}{8}$.

Let E_n be the set of all endpoints of intervals $J(F)$ for $F \in \hat{\mathcal{F}}_n$.

Let $(\varepsilon(1,\cdot), \delta(1,\cdot))$ be any pair of special α_1, β_1, θ_1 functions for $\hat{\mathcal{F}}_1$.

Now suppose that for each $k \leq n$ we have chosen a pair of special α_k, β_k, θ_k functions $(\varepsilon(k,\cdot), \delta(k,\cdot))$ for $\hat{\mathcal{F}}_k$ in such a way that

(i) whenever $1 \leq k \leq n - 1$, $e \in E_{k+1}$, $F \in \hat{\mathcal{F}}_k^2$, and $e \in F \cap J(F)^*$, then

$$\overline{S(e, \varepsilon(k+1, e), \theta_{k+1})} \cap H \subseteq B(F, \delta(k, F), \alpha_k, \beta_k);$$

(ii) whenever $1 \leq k \leq n - 1$, $e \in E_{k+1}$, and $e \in E_k$, then

$$\overline{S(e, \varepsilon(k+1, e), \theta_{k+1})} \cap H \subseteq S(e, \varepsilon(k, e), \theta_k);$$

(iii) whenever $1 \leq k \leq n - 1$, $K \in \hat{\mathcal{F}}_{k+1}^2$, $F \in \hat{\mathcal{F}}_k^2$, and $K \subseteq F$, then

$$\overline{B(K, \delta(k+1, K), \alpha_{k+1}, \beta_{k+1})} \cap H \subseteq B(F, \delta(k, F), \alpha_k, \beta_k).$$

Then we construct $(\varepsilon(n+1,\cdot), \delta(n+1,\cdot))$ as follows. Let (ε, δ) be any pair of special α_{n+1}, β_{n+1}, θ_{n+1} functions for $\hat{\mathcal{F}}_{n+1}$. If $e \in E_{n+1} - E_n$, then for some unique $F \in \hat{\mathcal{F}}_n^2$, $e \in F \cap J(F)^*$, so by (12) we can choose $\xi(e) > 0$ such that $\eta \leq \xi(e)$ implies

$$\overline{S(e, \eta, \theta_{n+1})} \cap H \subseteq B(F, \delta(n, F), \alpha_n, \beta_n).$$

We set $\varepsilon(n+1, e) = \min\{\varepsilon(e), \xi(e)\}$. On the other hand, if $e \in E_{n+1} \cap E_n$,

then we set $\varepsilon(n+1, e) = \min \{\varepsilon(e), \frac{1}{2}\varepsilon(n, e)\}$.

If $F \in \hat{\mathcal{F}}^2_{n+1}$, then there exists a unique $K \in \hat{\mathcal{F}}^2_n$ with $F \subseteq K$. Set

$$\delta(n+1, F) = \min \{\delta(F), \frac{1}{2}\delta(n, K)\}.$$

By Lemma 14, $(\varepsilon(n+1,\cdot), \delta(n+1,\cdot))$ is a pair of special α_{n+1}, β_{n+1}, θ_{n+1} functions for $\hat{\mathcal{F}}_{n+1}$, and by (13) and (9), conditions (i), (ii), and (iii) are still satisfied when n is replaced by n+1. Thus we can inductively construct a pair $(\varepsilon(n,\cdot), \delta(n,\cdot))$ of special α_n, β_n, θ_n functions for $\hat{\mathcal{F}}_n$ in such a way that conditions (i), (ii) and (iii) are satisfied for every n.

Let

$$U_n = \left[\bigcup_{e \in E_n} S(e, \varepsilon(n, e), \theta_n)\right] \cup \left[\bigcup_{F \in \hat{\mathcal{F}}^2_n} B(F, \delta(n, F), \alpha_n, \beta_n)\right].$$

Then U_n is open. For fixed n, all the various sets $S(e, \varepsilon(n, e), \theta_n)$ $(e \in E_n)$ and $B(F, \delta(n, F), \alpha_n, \beta_n)$ $(F \in \hat{\mathcal{F}}^2_n)$ are open and pairwise disjoint, so that every component of U_n is contained in one of the sets $S(e, \varepsilon(n, e), \theta_n)$ $(e \in E_n)$ or $B(F, \delta(n, F), \alpha_n, \beta_n)$ $(F \in \hat{\mathcal{F}}^2_n)$. It therefore follows from (16) and (17) that if Ω is any component of U_n, then

(23) $$\overline{\Omega} \cap X \subseteq A_n.$$

From the fact that $(\varepsilon(n,\cdot), \delta(n,\cdot))$ is a pair of special α_n, β_n, θ_n functions for $\hat{\mathcal{F}}_n$ together with conditions (18) and (19), it follows that

$$\overline{U}_n \cap H = [\bigcup_{e \in E_n} \overline{S(e, \varepsilon(n, e), \theta_n)} \cap H] \cup$$

$$[\bigcup_{F \in \hat{\mathcal{F}}_n^2} \overline{B(F, \delta(n, F), \alpha_n, \beta_n)} \cap H].$$

Consequently, conditions (i), (ii), (iii), together with the fact that

$$e \in E_{n+1} - E_n \Rightarrow e \in F \cap J(F)^* \text{ for some } F \in \hat{\mathcal{F}}_n^2,$$

show that $\overline{U}_{n+1} \cap H \subseteq U_n$ for every n.

By Urysohn's Lemma, there exists a continuous function $g_n : H \to [0, 1]$ such that

$$g_n(z) = 1 \quad \text{for } z \in H - U_n$$

and

$$g_n(z) = 0 \quad \text{for } z \in \overline{U}_{n+1} \cap H.$$

Let $g(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} g_n(z)$. Then $0 \leq g(z) \leq 1$, and the series converges uniformly, so g is continuous on H.

If $z \in H - U_n$, then $z \in H - U_m$ for every $m \geq n$, so that $1 = g_n(z) = g_{n+1}(z) = g_{n+2}(z) = \ldots$, and hence

$$\text{(24)} \qquad g(z) \geq \sum_{m=n}^{\infty} \frac{1}{2^m} = \frac{1}{2^{n-1}} \qquad (z \in H - U_n).$$

Also, if $z \in U_{n+1}$, then $z \in U_1, U_2, \ldots, U_{n+1}$, so that

$$0 = g_1(z) = g_2(z) = \ldots = g_n(z), \text{ and}$$

$$\text{(25)} \qquad g(z) \leq \sum_{m=n+1}^{\infty} \frac{1}{2^m} = \frac{1}{2^n} \qquad (z \in U_{n+1}).$$

We assert that

(26) for each $x_o \in A$, $g(z) \to 0$ as $z \to x_o$ with $z \in S(x_o, 1, \frac{3\pi}{8})$.

Take any natural number n. Since $x_o \in A_{n+1} = \bigcup \hat{\mathcal{F}}_{n+1}$, either $x_o \in E_{n+1}$ or else $x_o \in F \cap J(F)^*$ for some $F \in \hat{\mathcal{F}}_{n+1}^2$. In the first

case, set $\eta = \varepsilon(n+1, x_0)$. In the second case, (12) shows that we can choose $\eta > 0$ so that

$$S(x_0, \eta, \tfrac{3\pi}{8}) \subseteq B\,(F, \delta(n+1, F), \alpha_{n+1}, \beta_{n+1}).$$

Suppose $\langle x, y\rangle \in S(x_0, 1, \frac{3\pi}{8})$ and $y < \eta$. Then, in the first case, $\langle x, y\rangle \in S(x_0, \eta, \frac{3\pi}{8}) \subseteq S(x_0, \varepsilon(n+1, x_0), \theta_{n+1}) \subseteq U_{n+1}$, and in the second case,

$\langle x, y\rangle \in S(x_0, \eta, \frac{3\pi}{8}) \subseteq B(F, \delta(n+1, F), \alpha_{n+1}, \beta_{n+1}) \subseteq U_{n+1}$. So, by (25),

$$(\langle x, y\rangle \in S(x_0, 1, \tfrac{3\pi}{8}) \text{ and } y < \eta) \Rightarrow \langle x, y\rangle \in U_{n+1}$$
$$\Rightarrow 0 \le g(x, y) \le \frac{1}{2^n}.$$

This proves (26).

Let x_0 be a point in X and γ any arc at x_0. Suppose $g(z) \to 0$ as $z \to x_0$ along γ. Then γ has a subarc γ' with one endpoint at x_0 such that $\gamma' - \{x_0\} \subseteq g^{-1}((-\frac{1}{2^n}, \frac{1}{2^n}))$. By (24), $\gamma' - \{x_0\} \subseteq U_n$. Therefore, by (23), $x_0 \in A_n$. Since n was arbitrary, $x_0 \in \bigcap_{n=1}^{\infty} A_n = A$. Thus,

(27) if there exists an arc γ at x_0 such that $g(z) \to 0$ as z approaches x_0 along γ, then $x_0 \in A$.

Now define

$$f(x, y) = g(x, y) \sin \frac{1}{y} + i\, g(x, y) \qquad (\langle x, y\rangle \in H).$$

If $x_0 \in A$, then, by (26), $f(z) \to 0$ as $z \to x_0$ with $z \in S(x_0, 1, \frac{3\pi}{8})$. Thus every point of A is in the set of curvilinear convergence of f.

Conversely, suppose x_0 is any point of the set of curvilinear convergence of f. Let γ be an arc at x_0 such that f approaches the limit $c + di$ along γ. Then g approaches the limit d along γ. If d

is different from zero, then $g(x, y) \sin \frac{1}{y}$ (the real part of f) cannot approach any limit along γ -- a contradiction. Therefore g approaches the limit 0 along γ, and, by (27), $x_0 \in A$. Therefore A is the set of curvilinear convergence of f. ■

5. Boundary Functions for Continuous Functions

Lemma 15. Let E be a metric space, Y a separable metric space. Suppose that $\varphi : E \to Y$ is a function having the following property. For every open set $U \subseteq Y$ there exists an F_σ set $L \subseteq E$ and a countable set $N \subseteq E$ such that

$$\varphi^{-1}(U) \subseteq L \subseteq \varphi^{-1}(\overline{U}) \cup N.$$

Then there exists a countable set $M \subseteq E$ such that $\varphi|_{E-M}$ is of class $(F_\sigma(E - M))$.

Proof. Let $\mathcal{B}$ be a countable base for Y. For each $B \in \mathcal{B}$, let $L(B) \subseteq E$ be an F_σ set and let $N(B) \subseteq E$ be a countable set such that

$$\varphi^{-1}(B) \subseteq L(B) \subseteq \varphi^{-1}(\overline{B}) \cup N(B)$$

Let $M = \bigcup_{B \in \mathcal{B}} N(B)$. Then M is countable. Let $E_0 = E - M$ and let $\varphi_0 = \varphi|_{E_0}$. We show that φ_0 is of class $(F_\sigma(E_0))$.

Let W be any open subset of Y. If $p \in W$, there exists $r > 0$ such that $S(r, p) \subseteq W$. Choose $B \in \mathcal{B}$ so that $p \in B \subseteq S(\frac{1}{2} r, p)$. Then $\overline{B} \subseteq S(r, p) \subseteq W$. It follows that

$$W = \bigcup_{B \in \mathcal{Q}(W)} B = \bigcup_{B \in \mathcal{Q}(W)} \overline{B},$$

where $\mathcal{Q}(W) = \{B \in \mathcal{B} : \overline{B} \subseteq W\}$. Therefore

$$\varphi_0^{-1}(W) = E_0 \cap \varphi^{-1}(W) = E_0 \cap \bigcup_{B \in \mathcal{Q}(W)} \varphi^{-1}(B)$$

$$\subseteq E_o \cap \bigcup_{B \in \mathcal{Q}(W)} L(B)$$

$$\subseteq E_o \cap \bigcup_{B \in \mathcal{Q}(W)} [\varphi^{-1}(\overline{B}) \cup N(B)]$$

$$\subseteq E_o \cap \bigcup_{B \in \mathcal{Q}(W)} [\varphi^{-1}(\overline{B}) \cup M]$$

$$= E_o \cap \bigcup_{B \in \mathcal{Q}(W)} \varphi^{-1}(\overline{B})$$

$$= E_o \cap \varphi^{-1}(W) = \varphi_o^{-1}(W).$$

Consequently $\varphi_o^{-1}(W) = E_o \cap \bigcup_{B \in \mathcal{Q}(W)} L(B)$, so $\varphi_o^{-1}(W)$ is of class $(F_\sigma(E_o))$. ■

Theorem 5. Let Y be a separable metric space and let $f : H \to Y$ be a continuous function. Suppose that $E \subseteq X$ and that $\varphi : E \to Y$ is a boundary function for f. Then there exists a countable set $M \subseteq E$ such that $\varphi|_{E-M}$ is of class $(F_\sigma(E - M))$.

Proof. Let U be any open subset of Y, and let $W = (\overline{U})'$. Let

$$E_n = \{x \in X : \text{there exists an arc } \gamma \text{ at } x, \text{ having one endpoint on } X_n, \text{ such that } \gamma - \{x\} \subseteq f^{-1}(U)\}$$

$$K = \{x \in X : \text{there exists an arc } \gamma \text{ at } x \text{ such that } \gamma - \{x\} \subseteq f^{-1}(W)\}.$$

Ob serve that

$$\varphi^{-1}(U) \subseteq \bigcup_{n=1}^{\infty} E_n,$$

and

$$\varphi^{-1}(W) \subseteq K.$$

For the time being, let n be a fixed natural number. For each $x \in K$ we can choose an arc γ_x at x such that

$$\gamma_x - \{x\} \subseteq H_n \cap f^{-1}(W).$$

Since an arc at x is by definition a simple arc, $\gamma_x - \{x\}$ is a connected set and hence must be contained within one nonempty component of $H_n \cap f^{-1}(W)$. Let U_x denote this component (for each $x \in K$).

Let T be the set of all points of K that are two-sided limit points of $\overline{E}_n$. We claim that if $x, y \in T$, then $x \neq y$ implies $U_x \cap U_y = \phi$. If $U_x \cap U_y \neq \phi$, then (since U_x and U_y are two components of the same set) U_x and U_y are equal. Let p be the endpoint of γ_x lying in U_x and let q be the endpoint of γ_y lying in $U_y = U_x$. We can join p and q by an arc γ lying in U_x. Putting γ_x, γ_y and γ together, we obtain an arc α with one endpoint at x and the other at y, such that $\alpha - \{x, y\} \subseteq U_x$. According to [12] we can choose a simple arc $\alpha' \subseteq \alpha$ having one endpoint at x and the other at y. Of course, $\alpha' - \{x, y\} \subseteq U_x \subseteq H_n \cap f^{-1}(W)$. Let I be the open interval in X with endpoints at x and y, and let $J = X - \overline{I}$. Let B be the bounded component of $H - \alpha'$ and let A be the other component. Since X_n is unbounded and does not meet α', $X_n \subseteq A$.

Because x is a two-sided limit point of $\overline{E}_n$, we can choose a point $w \in I \cap E_n$. Let β be an arc at w, having one endpoint on X_n, such that $\beta - \{w\} \subseteq f^{-1}(U)$. Then β does not meet α' (because $\alpha' - \{x, y\} \subseteq f^{-1}(W)$ and $f^{-1}(W) \cap f^{-1}(U) = \phi$), and therefore (since $\beta - \{w\}$ contains a point of $X_n \subseteq A$) $\beta - \{w\} \subseteq A$. It follows that $w \in \overline{A}$. This, however, is a contradiction, because the frontier of A (relative to the finite plane) is $\alpha' \cup J$. We conclude that, for $x,y \in T$, $x \neq y$ implies $U_x \cap U_y = \phi$.

An open set in the plane has only countably many components, so it follows that T must be countable. Let S be the set of all

points of $\overline{E}_n$ that are not two-sided limit points of $\overline{E}_n$. We know that S is countable, so

$$\begin{aligned} K \cap \overline{E}_n &= [K \cap (\overline{E}_n - S)] \cup [K \cap S] \\ &= T \cup [K \cap S] \end{aligned}$$

is countable.

Let $N = K \cap \bigcup_{n=1}^{\infty} \overline{E}_n = \bigcup_{n=1}^{\infty} (K \cap \overline{E}_n)$. Then N is countable, and, since $\varphi^{-1}(W) \subseteq K$,

$$\begin{aligned} \varphi^{-1}(U) &\subseteq E \cap \bigcup_{n=1}^{\infty} E_n \subseteq E \cap \bigcup_{n=1}^{\infty} \overline{E}_n \\ &= (E \cap K \cap \bigcup_{n=1}^{\infty} \overline{E}_n) \cup ((E - K) \cap \bigcup_{n=1}^{\infty} \overline{E}_n) \\ &\subseteq (E \cap N) \cup (E - K) \subseteq (E \cap N) \cup (E - \varphi^{-1}(W)) \\ &= (E \cap N) \cup \varphi^{-1}(\overline{U}). \end{aligned}$$

Thus $\varphi^{-1}(U) \subseteq E \cap \bigcup_{n=1}^{\infty} \overline{E}_n \subseteq (E \cap N) \cup \varphi^{-1}(\overline{U})$, and the desired result follows from Lemma 15. ■

Corollary. Let Y be either the Riemann sphere, the real line, or a finite-dimensional Euclidean space. If $f : H \to Y$ is a continuous function, if $E \subseteq X$, and if $\varphi : E \to Y$ is a boundary function for f, then φ is of honorary Baire class 2 (E, Y).

Next we show that the foregoing corollary is as strong as possible in the sense that if E is any subset of X and φ is a function of honorary Baire class 2 mapping E into a suitable space, then there exists a continuous function in H having φ as a boundary function. A proof of this result -- at least for real- or vector-valued functions -- was outlined by Bagemihl and Piranian [2, Theorem 8], in the case

where $E = X$. Although the construction given here is carried out much more explicitly than the construction given by Bagemihl and Piranian, my treatment differs from theirs in only two aspects that are of any significance. First of all, the proof of the theorem for arbitrary subsets E of X depends on Lemma 6 of the Introduction. Secondly, Bagemihl and Piranian say in the last line of their proof that there is "no difficulty now in extending f continuously to the whole of D in such a manner that ϕ is a boundary function for f." While this appears to be all right for real- or vector-valued functions, it is not clear why the extension should be so easy for functions taking values on the Riemann sphere. Theorem 7 of the present paper shows, however, that the result can be obtained for functions taking values on the sphere once it is known for vector-valued functions.

The following miniature closed graph theorem will be a convenience.

Lemma 16. Suppose that M is a metric space and that $u : M \to R$ is a function having the following properties:

(i) if $\{p_n\}$ is a convergent sequence of points of M, then $\{u(p_n)\}$ converges neither to $+\infty$ nor to $-\infty$;

(ii) if $\{p_n\} \subseteq M$, $p \in M$, and $y \in R$, and if $p_n \underset{n}{\to} p$ and $u(p_n) \underset{n}{\to} y$, then $u(p) = y$.

Then u is continuous.

Proof. Suppose that $\{p_n\}$ is a sequence of points in M converging to a point $p \in M$. Using (i) it is easy to show that $\{u(p_n)\}$ is a bounded sequence. Suppose that $\{u(p_n)\}$ does not converge to $u(p)$. Then there exists a subsequence $\{u(p_{n(k)})\}$ that converges to a real number $y \neq u(p)$. This, however, contradicts (ii). We conclude that

$u(p_n) \underset{n}{\to} u(p)$. ■

Lemma 17. Let $h : R \to R$ be a strictly increasing function such that $h(R)$ is neither bounded above nor bounded below. Then there exists a continuous weakly increasing function $h^* : R \to R$ such that $h^*(h(x)) = x$ for every $x \in R$.

Proof. Let $Z = h(R)$. Observe that $h^{-1} : Z \to R$ is strictly increasing. For any $x \in R$, the set $(-\infty, x] \cap Z$ is nonempty. Also, $h^{-1}((-\infty, x] \cap Z)$ is bounded above, because if we choose $y \in Z$ with $x \leq y$, then $h^{-1}((-\infty, x] \cap Z)$ is bounded above by $h^{-1}(y)$.

We claim that for every $x \in R$

$$\text{(27)} \qquad \sup h^{-1}((-\infty, x] \cap Z) = \sup h^{-1}((-\infty, x) \cap Z).$$

If $x \notin Z$, the equation is trivial. Suppose $x \in Z$. Then

$$y < h^{-1}(x) \Longrightarrow (h(y) < x \text{ and } h(y) \in Z),$$

so that $h((-\infty, h^{-1}(x))) \subseteq (-\infty, x) \cap Z$. Hence

$$(-\infty, h^{-1}(x)) \subseteq h^{-1}((-\infty, x) \cap Z),$$

so that $\sup h^{-1}((-\infty, x) \cap Z) \geq h^{-1}(x) = \sup h^{-1}((-\infty, x] \cap Z)$. The opposite inequality is trivial, so (27) is established.

We also claim that

$$\text{(28)} \qquad \inf h^{-1}((x, +\infty) \cap Z) = \sup h^{-1}((-\infty, x] \cap Z).$$

Obviously, $\inf h^{-1}((x, +\infty) \cap Z) \geq \sup h^{-1}((-\infty, x] \cap Z)$. Take any $y > \sup h^{-1}((-\infty, x] \cap Z)$. If $h(y) \leq x$, then $h(y) \in (-\infty, x] \cap Z$, and so $y \in h^{-1}((-\infty, x] \cap Z)$-- a contradiction. Thus $h(y) > x$ and $h(y) \in (x, +\infty) \cap Z$. Therefore $y \in h^{-1}((x, +\infty) \cap Z)$, and so $\inf h^{-1}((x, +\infty) \cap Z) \leq y$. In view of the choice of y, this implies

that

$$\inf h^{-1}((x, +\infty) \cap Z) \leq \sup h^{-1}((-\infty, x] \cap Z),$$

and (28) is established.

Define

$$h^*(x) = \sup h^{-1}((-\infty, x] \cap Z).$$

It is clear that h^* is weakly increasing and that $h^*(h(x)) = x$ for every real x. The continuity of h^* can easily be deduced from the equations

$$\sup h^*((-\infty, x)) = h^*(x)$$

$$\inf h^*((x, +\infty)) = h^*(x),$$

which are established as follows:

$$\begin{aligned}
\sup h^*((-\infty, x)) &= \sup_{y<x} \sup h^{-1}((-\infty, y] \cap Z) \\
&= \sup h^{-1}(\overline{(-\infty, x)} \cap Z) \\
&= \sup h^{-1}((-\infty, x] \cap Z) \\
&= h^*(x) \\
\inf h^*((x, +\infty)) &= \inf_{y>x} \sup h^{-1}((-\infty, y] \cap Z) \\
&= \inf_{y>x} \inf h^{-1}((y, +\infty) \cap Z) \\
&= \inf h^{-1}((x, +\infty) \cap Z) \\
&= \sup h^{-1}((-\infty, x] \cap Z) \\
&= h^*(x). \qquad \blacksquare
\end{aligned}$$

Theorem 6. Let E be any subset of X and let $\varphi: E \to R^q$ be any function of honorary Baire class $2(E, R^q)$. Then there exists a continuous function $f : H \to R^q$ such that φ is a boundary function for f.

Proof. Let $\psi : E \to R^q$ be a function of Baire class $1(E, R^q)$ and N a countable subset of E such that $\varphi(x) = \psi(x)$ for every $x \in E - N$. Let $\{s_n\}_{n=1}^{\infty}$ (with $n \neq m$ implying $s_n \neq s_m$) be a countable dense subset of X that includes every integer and every point of N. Let

$$t_n = 1 \qquad \text{if } s_n \text{ is an integer}$$

$$t_n = \frac{1}{2^n} \qquad \text{if } s_n \text{ is not an integer.}$$

Define

$$h(x) = \sum_{0 \leq s_n < x} t_n \qquad \text{if } x > 0$$

$$h(x) = -\sum_{x \leq s_n \leq 0} t_n \qquad \text{if } x \leq 0.$$

Then h is a strictly increasing function from R into R, and $\overline{h(R)}$ is bounded neither above nor below. Let h^* be the function described in Lemma 17.

Suppose that $0 < y < 1$. Then (for fixed x)

$$u - h^*\left(\frac{x - (1-y)u}{y} \right)$$

is a strictly increasing continuous function of u that approaches $+\infty$ as $u \to +\infty$ and $-\infty$ as $u \to -\infty$. Consequently there exists precisely one number $u(x, y)$ that satisfies the equation

$$u(x, y) - h^*\left(\frac{x - (1-y)u(x, y)}{y} \right) = 0. \tag{29}$$

I claim that $u(x, y)$ is a continuous function on $H_1 = \{\langle x, y\rangle : x, y \in R \text{ and } 0 < y < 1\}$. Suppose $\{\langle x_n, y_n\rangle\} \subseteq H_1$ and $\langle x_n, y_n\rangle \to \langle x, y\rangle \in H_1$. If $u(x_n, y_n) \to +\infty$, then

$$\frac{x_n - (1-y_n)u(x_n, y_n)}{y_n} \underset{n}{\to} -\infty,$$

and hence

$$u(x_n, y_n) - h^*\left(\frac{x_n - (1-y_n)u(x_n, y_n)}{y_n} \right) \underset{n}{\to} +\infty,$$

which contradicts (29). Thus $u(x_n, y_n)$ cannot approach $+\infty$. A similar argument shows that $u(x_n, y_n)$ cannot approach $-\infty$. Now assume that $u(x_n, y_n) \to u_0 \in R$. Then, by (29),

$$0 = \lim_{n\to\infty} [u(x_n, y_n) - h^*(\frac{x_n - (1-y_n)u(x_n, y_n)}{y_n})]$$

$$= u_0 - h^*(\frac{x - (1-y)u_0}{y}),$$

so $u_0 = u(x, y)$. By Lemma 16, u is continuous.

From Lemma 6, there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of continuous functions mapping X into R^q such that $g_n(x) \underset{n}{\to} \psi(x)$ for each $x \in E$. For $n \geq 2$, define

$$f_0(x, y) = (yn(n+1) - n)g_n(u(x, y)) + ((n+1) - yn(n+1))g_{n+1}(u(x, y))$$
$$\text{when } \frac{1}{n+1} \leq y \leq \frac{1}{n}.$$

Then f_0 is continuous on $\overline{H}_2 \cap H$. By the Tietze extension theorem, we can assume that f_0 is defined and continuous on all of H. Let

$$r_n = \inf_{x>s_n} h(x)$$

$$\ell_n = \sup_{x<s_n} h(x)$$

$$v_n = \varphi(s_n) - \psi(s_n) \qquad \text{if } s_n \in N$$

$$v_n = 0 \qquad \text{if } s_n \notin N.$$

If x and y are real numbers, define $x \vee y = \max\{x, y\}$. For $\langle x, y\rangle \in H$, set

$$\Delta_n(x, y) =$$
$$[(1 - ny) \vee 0][(1 - \frac{1}{r_n - \ell_n} | r_n + \ell_n - 2s_n + 2\frac{s_n - x}{y} |) \vee 0]v_n.$$

Then Δ_n is continuous in H. Observe that $\Delta_n(x, y) = 0$ when $y \geq \frac{1}{n}$.

Using this fact, it is easy to show that, if we set

$$f = f_o + \sum_{n=1}^{\infty} \Delta_n,$$

then f is defined and continuous in H. We now show that φ is a boundary function for f.

Let p be any point of E. The line

(30) $$x = (h(p) - p)\, y + p$$

passes through (p, 0), and the part of it that lies in $\overline{H}_1$ is an arc at p. We will show that f approaches $\psi(p)$ along this line. If we substitute $(h(p) - p)y + p$ for x in the expression for $\Delta_n(x, y)$, we obtain

(31) $$\Delta_n(x, y) =$$
$$[(1 - ny) \vee 0][(1 - \frac{1}{r_n - \ell_n} |r_n + \ell_n + 2(\frac{1}{y} - 1)(s_n - p) - 2h(p)|) \vee 0]\ v_n.$$

If $p \leq s_n$, then $h(p) \leq \ell_n$, and one can verify directly that (31) vanishes. If $p > s_n$, then $h(p) \geq r_n$, and again one can verify directly that (31) vanishes. Thus $\Delta_n(x, y)$ vanishes along that part of the line (30) lying in H.

Solving (30) for h(p), we find that, along the given line,

$$h(p) = \frac{x - (1-y)p}{y},$$

and hence $p = h^*(h(p)) = h^*(\frac{x - (1-y)p}{y})$. Therefore, if $0 < y < 1$, $p = u(x, y)$. So, if $\langle x, y \rangle$ satisfies (30), $n \geq 2$, and $\frac{1}{n+1} \leq y \leq \frac{1}{n}$, then

$$f_o(x, y) = (yn(n+1) - n)g_n(p) + ((n+1) - yn(n+1))g_{n+1}(p).$$

Since the coefficients of $g_n(p)$ and $g_{n+1}(p)$ in the above expression add up to 1 and since both coefficients lie in $[0, 1]$, $f_o(x, y)$ lies on the line segment joining $g_n(p)$ to $g_{n+1}(p)$, and it follows that $f_o(x, y)$ approaches $\psi(p)$ as $\langle x, y\rangle$ approaches p along the line (30). Since each Δ_n vanishes on the part of this line lying in H, $f(x, y)$ also approaches $\psi(p)$ along the line.

Let s_m be any point of N. We show that f approaches $\varphi(s_m)$ along the part of the line

$$(32) \qquad x = \left(\frac{r_m + \ell_m}{2} - s_m\right)y + s_m$$

that lies in H. Again, we first consider the value of Δ_n along the given line. Substituting the value of x given by (32) into the expression for Δ_n, we obtain

$$(33) \qquad \Delta_n(x, y) =$$

$$[(1-ny) \vee 0]\left[\left(1 - \frac{1}{r_n-\ell_n}\left|r_n - r_m + \ell_n - \ell_m + 2\left(\frac{1}{y} - 1\right)(s_n - s_m)\right|\right) \vee 0\right]v_n.$$

If $s_m < s_n$, then $\ell_m < r_m \leq \ell_n < r_n$, and one can verify directly that (33) vanishes. If $s_n < s_m$, then $\ell_n < r_n \leq \ell_m < r_m$, and again one can verify that (33) vanishes. Thus, for $n \neq m$, $\Delta_n(x, y) = 0$ when $\langle x, y\rangle$ lies on the line (32) and in H.

If we take n = m in (33), we obtain

$$\Delta_m(x, y) = [(1 - my) \vee 0]v_m.$$

Therefore $\Delta_m(x, y)$ approaches $v_m = \varphi(s_m) - \psi(s_m)$ along the given line.

Take any $\langle x, y\rangle \in H_1$ satisfying (32), and take any a and b satisfying

$$(34) \qquad a < s_m < b.$$

Then $h(a) \leq \ell_m < \frac{r_m + \ell_m}{2} < r_m \leq h(b)$, so that $(h(a) - s_m)y + s_m < x < (h(b) - s_m)y + s_m$; from which we deduce that

$$h(a) < \frac{x - (1-y)s_m}{y} < h(b).$$

Since h^* is weakly increasing,

$$a = h^*(h(a)) \leq h^*(\frac{x - (1-y)s_m}{y}) \leq h^*(h(b)) = b.$$

Since a and b were taken to be any two numbers satisfying (34), we conclude that

$$s_m = h^*(\frac{x - (1-y)s_m}{y}),$$

whence it follows that $u(x, y) = s_m$. Thus

$$f_o(x, y) = (yn(n+1) - n)g_n(s_m) + ((n+1) - yn(n+1))g_{n+1}(s_m)$$

when $\langle x, y\rangle$ lies on the given line and $\frac{1}{n+1} \leq y \leq \frac{1}{n}$. Consequently $f_o(x, y)$ approaches $\psi(s_m)$ along the line (32). So f approaches $\psi(s_m) + \varphi(s_m) - \psi(s_m) = \varphi(s_m)$, and the theorem is proved. ■

Theorem 7. Let E be any subset of X and let $\varphi: E \to S^2$ be any function of honorary Baire class $2(E, S^2)$. Then there exists a continuous function $f : H \to S^2$ such that φ is a boundary function for f.

Proof. The proof of this theorem is very similar to that of Theorem 1. Since $S^2 \subseteq R^3$, there exists, by Theorem 6, a continuous function $g : H \to R^3$ having φ as a boundary function. Let

$$K = g^{-1}(\{v \in R^3 : |v| = \tfrac{1}{2}\})$$

$$L = g^{-1}(\{v \in R^3 : |v| \geq \tfrac{1}{2}\})$$

$$F = g^{-1}(\{v \in R^3 : |v| \leq \tfrac{1}{2}\}).$$

Let $g_o = g|_K$. H is homeomorphic to R^2, so by [5, Lemma 2.9, p. 299], g_o can be extended to a continuous function

$$g_1 : H \to \{v \in R^3 : |v| = \tfrac{1}{2}\}.$$

Define $f_1 : H \to R^3 - \{0\}$ by setting

$$f_1(z) = g(z) \qquad \text{if } z \in L$$
$$f_1(z) = g_1(z) \qquad \text{if } z \in F.$$

Then, since F and L are closed, f_1 is continuous on H. It is easy to verify that φ is a boundary function for f_1. Let $P_o : R^3 - \{0\} \to S^2$ be the 0-projection onto S^2 (see page 11), and let f be the composite function $P_o \circ f_1$. Then f maps H continuously into S^2, and $P_o \circ \varphi = \varphi$ is a boundary function for f. ■

CHAPTER II

BOUNDARY FUNCTIONS FOR DISCONTINUOUS FUNCTIONS

6. Boundary Functions for Baire Functions

It is not known whether the set of curvilinear convergence of a Borel-measurable function defined in H is necessarily a Borel set. The answer is not known even for functions of Baire class 1. However, a theorem on boundary functions that is similar to the corresponding result for continuous functions in H can be proved for functions of Baire class ξ in H.

Definition. If A and B are two sets, we will call A and B <u>equivalent</u> and write $A \simeq B$ if and only if $A - B$ and $B - A$ are both countable. It is easy to check that $\simeq$ is an equivalence relation.

Lemma 18. If $A \simeq E$, then $S - A \simeq S - E$ for any set S. If $A_n \simeq E_n$ for all n in some countable set N, then

$$\bigcup_{n \in N} A_n \simeq \bigcup_{n \in N} E_n \text{ and } \bigcap_{n \in N} A_n \simeq \bigcap_{n \in N} E_n.$$

The proof of this lemma is routine.

Definition. An interval of real numbers will be called <u>nondegenerate</u> if it contains more than one point.

Lemma 19. Any union of nondegenerate intervals is equivalent to an open set.

Proof. Let $\mathcal{J}$ be any family of nondegenerate intervals. It will suffice to prove that $\bigcup_{I\in\mathcal{J}} I - \bigcup_{I\in\mathcal{J}} I^*$ is countable. We can write

$$\bigcup_{I\in\mathcal{J}} I^* = \bigcup_n J_n,$$

where $\{J_n\}$ is a countable family of disjoint open intervals. If

$$x_o \in \bigcup_{I\in\mathcal{J}} I - \bigcup_{I\in\mathcal{J}} I^*,$$

then x_o is an endpoint of I_o for some $I_o \in \mathcal{J}$. For some n, $I_o^* \subseteq J_n$, so that $x_o \in \bar{J}_n$. But $x_o \notin J_n$, so x_o is an endpoint of J_n. Thus $\bigcup_{I\in\mathcal{J}} I - \bigcup_{I\in\mathcal{J}} I^*$ is contained in the set of all endpoints of the various J_n, and the lemma is proved. ■

Lemma 20. Let h be a weakly increasing real-valued function on a nonempty set $E \subseteq R$. Suppose that $|x - h(x)| \leq 1$ for every $x \in E$. Then h can be extended to a weakly increasing real-valued function h_1 on R.

Proof. Let $e = \inf E$ (e may be $-\infty$). For each $x \in (e, +\infty)$, set

$$h_1(x) = \sup h((-\infty, x] \cap E).$$

Since $|t - h(t)| \leq 1$ for each $t \in E$,

$$t \in (-\infty, x] \cap E \Rightarrow h(t) \leq x + 1,$$

so h_1 is finite-valued. If $e = -\infty$ we are done. If $e > -\infty$, then $x \in E$ implies $h(x) \geq x - 1 \geq e - 1$, so h is bounded below. For $x \in (-\infty, e]$ set

$$h_1(x) = \inf h(E).$$

It is easy to verify that h_1 has the required properties. ■

Lemma 21. Let Y be a metric space, $f : R \to Y$ a function of Baire class $\xi(R, Y)$, and suppose that $h : R \to R$ is weakly increasing. Then there exists a countable set $N \subseteq R$ such that the composite function $f \circ h|_{R-N}$ is of Baire class $\xi(R - N, Y)$.

Proof. Let N be the set of discontinuities of h. By a well-known theorem, N must be countable. But then $h|_{R-N}$ is continuous, so that $f \circ (h|_{R-N}) = (f \circ h)|_{R-N}$ is of Baire class $\xi(R - N, Y)$. ∎

Lemma 22. Let Y be a separable arcwise connected metric space, E any metric space, and let $\varphi : E \to Y$ be a function having the following property. For every open set $U \subseteq Y$ there exists a set $T \in P^{\xi+1}(E)$ such that $\varphi^{-1}(U) \subseteq T \subseteq \varphi^{-1}(\overline{U})$. Then, if $\xi \geq 2$, φ is of Baire class $\xi(E, Y)$.

Proof. The proof is similar to that of Lemma 15. Let $\mathcal{B}$ be a countable base for Y, and suppose that W is any open subset of Y. Let

$$\mathcal{Q}(W) = \{U \in \mathcal{B} : \overline{U} \subseteq W\}.$$

The argument in the proof of Lemma 15 shows that

$$W = \bigcup_{U \in \mathcal{Q}(W)} U = \bigcup_{U \in \mathcal{Q}(W)} \overline{U}.$$

For each $U \in \mathcal{B}$, let $T(U) \in P^{\xi+1}(E)$ be chosen so that $\varphi^{-1}(U) \subseteq T(U) \subseteq \varphi^{-1}(\overline{U})$. Then

$$\begin{aligned}\varphi^{-1}(W) &= \bigcup_{U \in \mathcal{Q}(W)} \varphi^{-1}(U) \subseteq \bigcup_{U \in \mathcal{Q}(W)} T(U) \\ &\subseteq \bigcup_{U \in \mathcal{Q}(W)} \varphi^{-1}(\overline{U}) = \varphi^{-1}(W).\end{aligned}$$

Thus $\varphi^{-1}(W) = \bigcup_{U \in \mathcal{Q}(W)} T(U)$, and since $P^{\xi+1}(E)$ is closed under countable unions, $\varphi^{-1}(W) \in P^{\xi+1}(E)$. Therefore φ is of Baire class $\xi(E, Y)$. ∎

Theorem 8. Let Y be a separable arcwise connected metric space, $f : H \to Y$ a function of Baire class $\xi(H, Y)$ where $\xi \geq 1$, E a subset of X, and $\varphi: E \to Y$ a boundary function for f. Then φ is of Baire class $\xi + 1(E, Y)$.

Proof. Let U be any open subset of Y and let $V = Y - \overline{U}$. Set

$$A = \varphi^{-1}(U) \qquad B = \varphi^{-1}(V)$$

$$C = A \cup B.$$

Observe that $A \cap B = \phi$. For each $x \in C$, choose an arc γ_x at x such that

$$\lim_{\substack{z \to x \\ z \in \gamma_x}} f(z) = \varphi(x)$$

$$\gamma_x \subseteq \{z : |z - x| \leq 1\}$$

$$\gamma_x - \{x\} \subseteq f^{-1}(U) \qquad \text{if } x \in A$$

$$\gamma_x - \{x\} \subseteq f^{-1}(V) \qquad \text{if } x \in B.$$

Notice that if $x \in A$ and $y \in B$, then $\gamma_x \cap \gamma_y = \phi$.

We will say that γ_x meets γ_y in $\overline{H}_n$ provided that γ_x and γ_y have subarcs γ_x' and γ_y' respectively such that $x \in \gamma_x' \subseteq \overline{H}_n$, $y \in \gamma_y' \subseteq \overline{H}_n$, and $\gamma_x' \cap \gamma_y' \neq \phi$. Let

$$L_a = \{x \in A : (\forall n)(\exists y)(y \in C,\ y \neq x, \text{ and } \gamma_y \text{ meets } \gamma_x \text{ in } \overline{H}_n)\}$$

$$L_b = \{x \in B : (\forall n)(\exists y)(y \in C,\ y \neq x, \text{ and } \gamma_y \text{ meets } \gamma_x \text{ in } \overline{H}_n)\}$$

$$M_a = \{x \in A : (\exists n)(\gamma_x \text{ meets no } \gamma_y \text{ (with } y \neq x) \text{ in } \overline{H}_n)\}$$

$$M_b = \{x \in B : (\exists n)(\gamma_x \text{ meets no } \gamma_y \text{ (with } y \neq x) \text{ in } \overline{H}_n)\}.$$

$$L = L_a \cup L_b$$

$$M = M_a \cup M_b.$$

Observe that L_a, L_b, M_a, M_b are pairwise disjoint, and that $A = L_a \cup M_a$ and $B = L_b \cup M_b$.

For each $x \in M$, let $n(x)$ be a positive integer such that γ_x meets no γ_y (with $y \neq x$) in $\overline{H}_{n(x)}$. Then $n \geq n(x)$ implies that γ_x meets no γ_y in $\overline{H}_n$. Let

$$K_n = \{x \in C : \gamma_x \text{ meets } X_n, \text{ and, if } x \in M, n \geq n(x)\}.$$

Then $K_n \subseteq K_{n+1}$ for each n, and $C = \bigcup_{n=1}^{\infty} K_n$.

We next show that for each positive integer n and each $x \in L_a$ there exists a nondegenerate closed interval I_x^n such that $x \in I_x^n \subseteq L_a \cup (X - K_n)$. By the definition of L_a, there exists $y \in C$ $(y \neq x)$ such that γ_y meets γ_x in $\overline{H}_n$. Let I_x^n be the closed interval having its endpoints at x and y. Let t be any point of I_x^n. We must prove that $t \in L_a \cup (X - K_n)$. If $t \notin K_n$, we are done. So assume $t \in K_n$. Then γ_t meets X_n, and hence it is clear from Figure 5 that γ_t must meet either γ_x or γ_y in $\overline{H}_n$. (This argument can be rigorized by means of Theorem 11.8 on p. 119 in [11].) But, if $t \in M$, then (because $t \in K_n$) $n \geq n(t)$, so that this situation is impossible. Therefore $t \notin M$. Now $x \in L_a \subseteq A$, so, since γ_x intersects γ_y, $y \notin B$. Hence $y \in C - B = A$. Similarly, since γ_t intersects γ_x or γ_y, $t \in C - B = A$. Thus $t \in A - M = L_a$, and we have shown that $I_x^n \subseteq L_a \cup (X - K_n)$.

Let $W_n = \bigcup_{x \in L_a} I_x^n$. For each n,

$$L_a \subseteq W_n \cap C \subseteq [L_a \cup (X - K_n)] \cap C,$$

and therefore

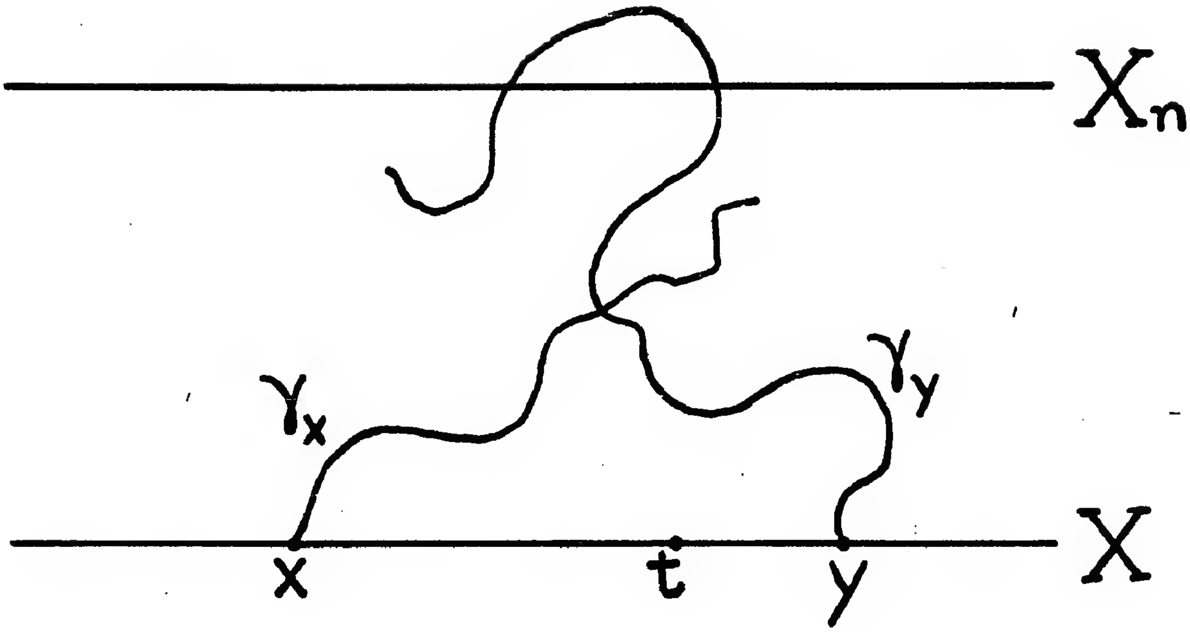

Figure 5.

$$L_a \subseteq \left(\bigcap_{n=1}^{\infty} W_n\right) \cap C$$

$$\left\{\bigcap_{n=1}^{\infty} [L_a \cup (X - K_n)]\right\} \cap C$$

$$= \left[L_a \cup \left(X - \bigcup_{n=1}^{\infty} K_n\right)\right] \cap C$$

$$= (L_a \cap C) \cup \left(C - \bigcup_{n=1}^{\infty} K_n\right) = L_a \cup \phi = L_a.$$

It follows that $L_a = \left(\bigcap_{n=1}^{\infty} W_n\right) \cap C$. By Lemma 19, each W_n is equivalent to an open set, so there exists a G_δ set $G_a \subseteq X$ such that

$$L_a \simeq G_a \cap C.$$

A similar argument shows that there exists a G_δ set $G_b \subseteq X$ such that

$$L_b \simeq G_b \cap C.$$

Next we study the properties of M_a. In doing this, it is convenient to define a function $\pi : R^2 \to R$ by setting $\pi(x, y) = x$. If $x \in M \cap K_n$, then, starting at x and proceeding along γ_x, let $p_n(x)$ be the first point of X_n that is reached. Define $h_n^o : M \cap K_n \to R$ by setting $h_n^o(x) = \pi(p_n(x))$. If $x, x' \in M \cap K_n$ and $x < x'$, then, since γ_x cannot meet $\gamma_x{}'$ in $\overline{H}_n$, it is evident that $p_n(x)$ must lie to the left of $p_n(x')$; that is, $\pi(p_n(x)) < \pi(p_n(x'))$. Thus h_n^o is a strictly increasing function on $M \cap K_n$. Moreover,

$$|x - h_n^o(x)| \le 1 \quad \text{because } \gamma_x \subseteq \{z : |z - x| \le 1\}.$$

So, by Lemma 20, h_n^o can be extended to a weakly increasing function $h_n : X \to R$. Let

$$g_n(x) = f\left(h_n(x), \frac{1}{n}\right) \qquad (x \in R).$$

$f(x, \frac{1}{n})$ is a function (of x) of Baire class $\xi(X, Y)$, so, by Lemma 21, there exists a countable set N_n X such that $g_n|_{X-N_n}$ is of Baire class $\xi(X - N_n, Y)$. Let $N = \bigcup_{n=1}^{\infty} N_n$. Then $g_n|_{M-N}$ is of Baire class $\xi(M - N, Y)$.

For $x \in M \cap K_n$, $g_n(x) = f(h_n^o(x), \frac{1}{n}) = f(p_n(x))$. If $x \in M$, then for all sufficiently large n, $x \in M \cap K_n$, so

$$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f(p_n(x)) = \varphi(x).$$

Thus $g_n|_M \underset{n}{\to} \varphi|_M$, so $g_n|_{M-N} \underset{n}{\to} \varphi|_{M-N}$, hence $\varphi|_{M-N}$ is of Baire class $\xi + 1(M - N, Y)$. It follows that there exists $D \in P^{\xi+2}(X)$ such that

$$A \cap (M - N) = (\varphi|_{M-N})^{-1}(U) = D \cap (M - N).$$

Obviously $A \cap M \simeq D \cap M$. Now,

$$L = L_a \cup L_b \simeq (G_a \cap C) \cup (G_b \cap C) = (G_a \cup G_b) \cap C,$$

so

$$\begin{aligned} M_a &= A \cap M \simeq D \cap M = D \cap (C - L) \\ &\simeq D \cap [C - ((G_a \cup G_b) \cap C)] \\ &= D \cap [X - (G_a \cup G_b)] \cap C. \end{aligned}$$

G_a and G_b are G_δ, so $X - (G_a \cup G_b)$ is F_σ, and hence

$$X - (G_a \cup G_b) \in P^2(X) \subseteq P^{\xi+2}(X).$$

Therefore $M_a \simeq F \cap C$, where $F \in P^{\xi+2}(X)$. Now, $G_a \in G_\delta(X) = Q^2(X)$, and since $\xi \geq 1$, $Q^2(X) \subseteq P^{\xi+2}(X)$, so $G_a \cup F \in P^{\xi+2}(X)$. But

$$A = L_a \cup M_a \simeq (G_a \cap C) \cup (F \cap C) = (G_a \cup F) \cap C,$$

so $A \simeq S \cap C$, where $S \in P^{\xi+2}(X)$. Since every countable set is F_σ, it is now easy to show that

$$A = T \cap C$$

for some $T \in P^{\xi+2}(X)$. From the definition of C it follows that $T \subseteq X - B$. Thus we have

$$\varphi^{-1}(U) = A \subseteq T \cap E \subseteq E - B = E - \varphi^{-1}(V) = \varphi^{-1}(\overline{U}).$$

$T \cap E \in P^{\xi+2}(E)$, so Lemma 22 shows that φ is of Baire class $\xi + 1(E, Y)$. ■

Corollary. Let Y be a separable arcwise-connected metric space, $f : H \to Y$ a Borel-measurable function, E a subset of X, and $\varphi : E \to Y$ a boundary function for f. Then φ is Borel-measurable.

Proof. f is of some Baire class $\xi(H, Y)$, hence φ is of Baire class $\xi + 1(E, Y)$, hence φ is Borel-measurable. ■

This corollary raises the question of whether a boundary function for a Lebesgue-measurable function is necessarily Lebesgue-measurable, which we answer in the next section.

7. Boundary Functions for Lebesgue-Measurable Functions

Suppose that a_0, b_0, a_1, b_1 are extended real numbers, and that $a_0 \leq b_0$, $a_1 \leq b_1$. To make the formalism more convenient we let $(-\infty) - (-\infty) = 0$ and $(+\infty) - (+\infty) = 0$. In other respects we adhere to the usual conventions regarding arithmetic operations that involve $-\infty$ or $+\infty$. Let

$$T(a_0, b_0; a_1, b_1) = \{ \langle x, y \rangle : 0 \leq y \leq 1 \text{ and } (a_1 - a_0)y + a_0 \leq x \leq (b_1 - b_0)y + b_0 \}.$$

A set of this form will be called a <u>closed</u> <u>trapezoid</u>. We also consider ϕ to be a closed trapezoid. A set S will be called a <u>trapezoid</u> if there exists a closed trapezoid T such that $T^1 \subseteq S \subseteq T$,

where T^1 denotes the interior of T relative to $\overline{H}_1$. Every trapezoid is Lebesgue-measurable, though not necessarily Borel-measurable.

If s, s' are disjoint line segments having endpoints $\langle a_o, 0\rangle$, $\langle a_1, 1\rangle$, and $\langle a_o', 0\rangle$, $\langle a_1', 1\rangle$ respectively, where $a_i \leq a_i'$ $(i = 0, 1)$, then let

$$T(s, s') = T(s', s) = T(a_o, a_o'; a_1, a_1').$$

If $s = s'$, then we let $T(s, s') = T(s', s) = s$. In what follows we will use the symbol X_o as an alternative designation for the x-axis X. This will enable us to make statements about X_i $(i = 0, 1)$ (where X_1 denotes, as before, $\{\langle x, 1\rangle : x \in R\}$).

We omit the proofs of the following two routine lemmas.

Lemma 23. Let theline segments s_1, s_2, s_3, s_4 each have one endpoint on X_o and the other on X_1, and assume that $i \neq j$ implies that either $s_i \cap s_j = \phi$ or $s_i = s_j$. If $T(s_1, s_2) \cap T(s_3, s_4) \neq \phi$, then

$$T(s_1, s_3) \subseteq T(s_1, s_2) \cup T(s_3, s_4).$$

Lemma 24. Let $\mathcal{S}$ be any set of line segments, each of which has one endpoint on X_o and the other on X_1, and no two of which intersect. Then $\bigcup_{s,s' \in \mathcal{S}} T(s, s')$ is a trapezoid.

Let m denote two-dimensional Lebesgue measure in R^2. If E is a measurable subset of some line in R^2, let $m^\ell(E)$ denote the linear Lebesgue measure of E. Let m_e and m_e^ℓ denote two-dimensional exterior measure and linear exterior measure, respectively; i.e., for any $E \subseteq R^2$,

$$m_e(E) = \inf\{m(U) : E \subseteq U \text{ and } U \text{ is open}\};$$

and if E is a subset of a line L, then

$m_e^{\ell}(E) = \inf \{m^{\ell}(U) : E \subseteq U \subseteq L$ and U is open relative to L}.

Theorem 9. Let $\mathcal{L}$ be any set of line segments, each of which has one endpoint on X_0 and the other on X_1, and no two of which intersect. Let $S = \bigcup \mathcal{L}$. Then

$$m_e(S) = \frac{1}{2}(m_e^{\ell}(S \cap X_0) + m_e^{\ell}(S \cap X_1)).$$

Proof. We may assume that $\mathcal{L}$ is nonempty. Let ε be any positive number. Choose an open set $U \subseteq R^2$ such that $S \subseteq U$ and

$$m(U) \leq m_e(S) + \varepsilon.$$

Let $E_i = S \cap X_i$ $(i = 0, 1)$. Choose sets $G_i \subseteq X_i$ that are open relative to X_i such that $E_i \subseteq G_i$ and

$$m^{\ell}(G_i) \leq m_e^{\ell}(E_i) + \varepsilon \qquad (i = 0, 1).$$

Let V be the union of all lines $L \subseteq R^2$ such that L meets both G_0 and G_1. It is easy to show that V is an open set. Furthermore, $S \subseteq V$ and $V \cap X_i = G_i$ $(i = 0, 1)$. Now let $W = U \cap V$. Then

$$W \text{ is open}, \quad S \subseteq W \subseteq U, \text{ and}$$

$$E_i \subseteq W \cap X_i \subseteq G_i \qquad (i = 0, 1).$$

If $s, s' \in \mathcal{L}$, define $s \equiv s'$ if and only if $T(s, s') \subseteq W$. It is easy to verify by means of Lemma 23 that $\equiv$ is an equivalence relation. Let Γ be the set of all equivalence classes. We prove that Γ is countable.

If $s \in \mathcal{L}$, we let $\langle a_i(s), i\rangle$ be the endpoint of s on X_i $(i = 0, 1)$. Then

$$s = \{\langle x, y\rangle \in R^2 : 0 \leq y \leq 1 \text{ and } x = (a_1(s) - a_0(s))y + a_0(s)\}.$$

Since s is compact and contained in W, there is no difficulty in showing that there exists $\delta_s > 0$ such that

$$\{\langle x, y\rangle \in R^2 ; 0 \leq y \leq 1 \quad \text{and}$$
$$(a_1(s) - a_0(s))y + a_0(s) - \delta_s < x < (a_1(s) - a_0(s))y + a_0(s) + \delta_s\} \subseteq W.$$

Let $J_i(s) = (a_i(s) - \delta_s, a_i(s) + \delta_s)$ $(i = 0, 1)$. A sketch will rapidly convince the reader that if $s, s' \in \mathcal{L}$, $J_0(s) \cap J_0(s') \neq \phi$, and $J_1(s) \cap J_1(s') \neq \phi$, then $T(s, s') \subseteq W$, so that $s \equiv s'$. Thus

$$(J_0(s) \times J_1(s)) \cap (J_0(s') \times J_1(s')) \neq \phi \Longrightarrow s \equiv s'.$$

For each $C \in \Gamma$, choose $s(C) \in C$ and let

$$Q(C) = J_0(s(C)) \times J_1(s(C)).$$

Then $C_1 \neq C_2 \Longrightarrow Q(C_1) \cap Q(C_2) = \phi$. Since each $Q(C)$ is a nonempty open subset of R^2, this implies that Γ is countable.

If $C \in \Gamma$, let

$$T(C) = \bigcup_{s, s' \in C} T(s, s').$$

By Lemma 24, $T(C)$ is a trapezoid. Also,

$$(35) \qquad C \subseteq T(C) \subseteq W.$$

Suppose that $C_1, C_2 \in \Gamma$ and $C_1 \neq C_2$. We claim that $T(C_1) \cap T(C_2) = \phi$. Assume that $T(C_1) \cap T(C_2) \neq \phi$. Then there exist $s_1, s_1' \in C_1$ and $s_2, s_2' \in C_2$ such that $T(s_1, s_1') \cap T(s_2, s_2') \neq \phi$. By Lemma 22,

$$T(s_1, s_2) \subseteq T(s_1, s_1') \cup T(s_2, s_2') \subseteq W,$$

so that $s_1 \equiv s_2$; a contradiction. Therefore $T(C_1) \cap T(C_2) = \phi$.

Let $K_i(C) = T(C) \cap X_i$ $(i = 0, 1)$. Then $K_i(C)$ is an interval and

$$(36) \qquad E_i \subseteq \bigcup_{C \in \Gamma} K_i(C) \subseteq W \cap X_i \subseteq G_i \qquad (i = 0, 1).$$

Furthermore, $C_1 \neq C_2$ implies that $K_i(C_1) \cap K_i(C_2) = \phi$. Using the formula for the area of a trapezoid, we find that

$$\frac{1}{2} [m^\ell(\bigcup_{C \in \Gamma} K_o(C)) + m^\ell(\bigcup_{C \in \Gamma} K_1(C))]$$

$$= \sum_{C \in \Gamma} \frac{1}{2} (m^\ell(K_o(C)) + m^\ell(K_1(C)))$$

$$= \sum_{C \in \Gamma} m(T(C)) = m(\bigcup_{C \in \Gamma} T(C)).$$

Let $\alpha = \frac{1}{2} [m^\ell(\bigcup_{C \in \Gamma} K_o(C)) + m^\ell(\bigcup_{C \in \Gamma} K_1(C))]$

$$= m(\bigcup_{C \in \Gamma} T(C)).$$

According to (35), $S \subseteq \bigcup_{C \in \Gamma} T(C) \subseteq W \subseteq U$, so that

$$(37) \qquad m_e(S) \leq \alpha \leq m(U) \leq m_e(S) + \varepsilon.$$

By (36),

$$(38) \qquad \frac{1}{2} (m_e^\ell(E_o) + m_e^\ell(E_1)) \leq \alpha \leq \frac{1}{2} (m^\ell(G_o) + m^\ell(G_1))$$
$$\leq \frac{1}{2} (m_e^\ell(E_o) + m_e^\ell(E_1)) + \varepsilon.$$

Since ε is arbitrary, inequalities (37) and (38) imply that

$$m_e(S) = \frac{1}{2} (m_e^\ell(E_o) + m_e^\ell(E_1)). \blacksquare$$

One wonders to what extent a result resembling the foregoing theorem might be obtainable without the hypothesis that no two of the line segments intersect. The following example is relevant to this question. Let M_o be a residual set of measure zero in X_o and let M_1

be a residual set of measure zero in X_1. Let $\langle x_o, y_o\rangle$ be any point of H_1. We claim that there is a line segment passing through $\langle x_o, y_o\rangle$ that has one endpoint in M_o and the other in M_1. For $\theta \in (0, \pi)$, let

$$F_1(\theta) = \langle (1 - y_o) \operatorname{ctn} \theta + x_o, 1\rangle \qquad \text{and}$$
$$F_o(\theta) = \langle x_o - y_o \operatorname{ctn} \theta, 0\rangle.$$

Then F_i is a homeomorphism of $(0, \pi)$ onto X_i, so $F_o^{-1}(M_o)$ and $F_1^{-1}(M_1)$ are both residual sets in $(0, \pi)$. Choose $\alpha \in F_o^{-1}(M_o) \cap F_1^{-1}(M_1)$. Let L be the line whose equation is

$$x = x_o + (y - y_o) \operatorname{ctn} \alpha.$$

Then L passes through the points $\langle x_o, y_o\rangle$, $F_o(\alpha)$ and $F_1(\alpha)$, so that $L \cap \overline{H}_1$ is the desired line segment. Let $\mathcal{L}$ be the set of all line segments having one endpoint in M_o and the other in M_1. Let $S = \bigcup \mathcal{L}$. Then $S \cap X_o$ and $S \cap X_1$ both have measure zero, but, as we have just shown, $H_1 \subseteq S$, so that S has infinite measure. See Problem 5 at the end of this paper.

Lemma 25. For every $\varepsilon > 0$ there exists a strictly increasing real-valued function h on R such that $h(R)$ has measure zero, and, for every real x, $|x - h(x)| \leq \varepsilon$.

Proof. For each integer n, let $I_n = [n\varepsilon, (n + 1)\varepsilon]$. Then $\bigcup_{n=-\infty}^{+\infty} I_n = R$. There exists a strictly increasing function $f : [0, 1] \to [0, 1]$ such that $m^{\ell}(f([0,1])) = 0$. For example, such a function may be defined as follows. Any number in $[0, 1)$ may be written in the form

$$.a_1a_2a_3\ldots a_n\ldots \qquad \text{(binary decimal)},$$

where the decimal does not end in an infinite unbroken string of 1's.

Set

$$f(\cdot a_1a_2a_3\ldots a_n\ldots) = \cdot b_1b_2b_3\ldots b_n\ldots \qquad \text{(ternary decimal)},$$

where $b_i = 0$ if $a_i = 0$ and $b_i = 2$ if $a_i = 1$.

Set $f(1) = 1$. Then f maps $[0, 1]$ into the Cantor ternary set, so $m^\ell(f([0, 1])) = 0$. It is easily shown that f is strictly increasing.

For each n, it is easy to obtain from f a function $f_n : I_n \to I_n$ such that f_n is strictly increasing and $m^\ell(f_n(I_n)) = 0$. Set

$$h(x) = f_n(x) \quad \text{for} \quad x \in (n\varepsilon, (n + 1)\varepsilon].$$

There is no difficulty in proving that h has the required properties. ■

Theorem 10. There exists an indexed family $\{\gamma_x\}_{x \in X}$ of simple arcs such that

(i) for each $x \in X$, γ_x is an arc at x

(ii) $x \neq y \Longrightarrow \gamma_x \cap \gamma_y = \phi$

(iii) $\bigcup_{x \in X} \gamma_x$ is a set of measure zero.

Proof. For each natural number n, let $h_n : R \to R$ be a strictly increasing function such that $h_n(R)$ has measure zero and, for every x, $|x - h_n(x)| \leq \frac{1}{n}$. For every $x \in R$, let $s_n(x)$ be the line segment joining the point $\left\langle h_n(x), \frac{1}{n} \right\rangle$ to the point $\left\langle h_{n+1}(x), \frac{1}{n+1} \right\rangle$. Since

$$h_n(x_1) < h_n(x_2) \Longrightarrow x_1 < x_2 \Longrightarrow h_{n+1}(x_1) < h_{n+1}(x_2),$$

we see that $x_1 \neq x_2$ implies $s_n(x_1) \cap s_n(x_2) = \phi$. Let $S_n = \bigcup\{s_n(x) : x \in R\}$. Then

$$S_n \cap X_n \subseteq \{\left\langle x, \tfrac{1}{n} \right\rangle : x \in h_n(R)\}$$

and
$$S_n \cap X_{n+1} \subseteq \{\left\langle x, \tfrac{1}{n+1} \right\rangle : x \in h_{n+1}(R)\},$$

so $m^\ell(S_n \cap X_n) = m^\ell(S_n \cap X_{n+1}) = 0$. It is easy to deduce from

Theorem 9 that

$$m_e(S_n) = \left(\frac{1}{n} - \frac{1}{n+1}\right) \frac{1}{2} \left(m_e^{\ell}(S_n \cap X_n) + m_e^{\ell}(S_n \cap X_{n+1})\right) = 0.$$

For $x \in X$, let $\gamma_x = \{x\} \cup \bigcup_{n=1}^{\infty} s_n(x)$. Since $\left\langle h_n(x), \frac{1}{n} \right\rangle \underset{n}{\to} x$, γ_x is an arc at x.

$$m_e\left(\bigcup_{x \in X} \gamma_x\right) \leq m_e(X) + m_e\left(\bigcup_{n=1}^{\infty} S_n\right)$$

$$\leq m_e(X) + \sum_{n=1}^{\infty} m_e(S_n) = 0,$$

so $\bigcup_{x \in X} \gamma_x$ is a set of measure zero. ■

Corollary. Let φ be an arbitrary function mapping X into any topological space Y having an element called 0. Then there exists a function $f : H \to Y$ such that $f(z) = 0$ almost everywhere and φ is a boundary function for f.

Proof. If $\{\gamma_x\}_{x \in X}$ is the family of arcs described in Theorem 10, let

$$f(z) = 0 \qquad \text{if } z \text{ is in no } \gamma_x$$

$$f(z) = \varphi(x) \qquad \text{if } z \in \gamma_x.$$

Then f is the desired function. ■

Corollary. There exists a real-valued Lebesgue-measurable function f defined in H having a nonmeasurable boundary function defined on X.

SOME UNSOLVED PROBLEMS

1. If A is an arbitrary set of type $F_{\sigma\delta}$ in X, does there necessarily exist a real-valued continuous function f defined in H having A as its set of curvilinear convergence? If φ is an arbitrary real-valued function of honorary Baire class 2 on A does there exist a continuous real-valued function f defined in H having A as its set of curvilinear convergence and φ as a boundary function?

2. (First proposed by J. E. McMillan [10]). If A is any set of type $F_{\sigma\delta}$ in X and if φ is any function of honorary Baire class 2(A, S^2), does there necessarily exist a continuous function $f : H \to S^2$ having A as its set of curvilinear convergence and φ as a boundary function?

3. If f is a real-valued Borel-measurable function defined in H, is the set of curvilinear convergence of f necessarily a Borel set? What if f is assumed to be of Baire class 1?

4. Let $S = \{\langle x, y, z\rangle \in R^3 : z > 0\}$. If f is a function defined in S, we define the set of curvilinear convergence of f in the obvious way. If f is continuous, is its set of curvilinear convergence necessarily a Borel set? Is it necessarily of type $F_{\sigma\delta}$?

5. Let $\mathcal{L}$ be a set of line segments each having one endpoint on X_o and the other on X_1, and let $S = \bigcup \mathcal{L}$. Assume that S is a Borel set. If $m^{\ell}(S \cap X_o)$ and $m^{\ell}(S \cap X_1)$ are known, what lower bound can be given

for m(S)? A solution to this problem might be helpful in attacking a problem of Bagemihl, Piranian, and Young [3, Problem 1].

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